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BUCKLING OF SHELLS OF REVOLUTION WITH VARIOUS WALL CONSTRUCTIONS

Volume 2 - Basic Equations and Method of Solution

by D. Bushnell, B. O. Almroth, and L. H. Sobel

Prepared by
LOCKHEED AIRCRAFT CORPORATION
Sunnyvale, Calif.
for Langley Research Center

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FOREWORD

This is the second of three volumes of a final report entitled "Buckling of Shells of Revolution with Various Wall Constructions". The three volumes have the following titles:

- Vol. 1 Numerical Results
- Vol. 2 Basic Equations and Method of Solution
- Vol. 3 User's Manual for BØSØR

The work described in these volumes was carried out under Contract NAS 1-6073 with the National Aeronautics and Space Administration.

ABSTRACT

Volume 1

Volume 1 presents the results of a parameter study performed with the computer program BØSØR (Buckling Of Shells Of Revolution) which is described in Volume 3. The axisymmetric collapse and the nonsymmetric bifurcation buckling behavior is studied for cylinders, cones, and spherical and toroidal shell segments subjected to axial compressive loads. Particular emphasis is placed on the effects of eccentricity in load application and on the influence of elastic end rings.

Volume 2

Volume 2 presents the equations on which the computer program BØSØR is based, as well as the method of solution of the equations. In addition, a set of more general stability equations is given in an appendix.

Volume 3

Volume 3 presents a comprehensive computer program (BØSØR) for the analysis of shells of revolution with axisymmetric loading. The program includes nonlinear prebuckling effects and is very general with respect to geometry of meridian, shell wall design, edge conditions, and loading. Despite its generality the program is easy to use. Branches are provided such that for commonly occurring cases the input data involves only basic information such as geometrical and material properties. The computer program has been verified by comparisons with other known solutions. The cards and a computer listing for this program are available from COSMIC, University of Georgia, Athens, Georgia, 30601.

NOTATION

A_{ij}	see Eq. (41)
a_i	see Eqs. (53), (55) and (62)
B_i	see Eqs. (52) and (54)
B_{33}, B_{66}	see Eq. (41)
BA11, etc.	boundary condition coefficients at A, see Eqs. (43) and (74)
BB11, etc.	boundary condition coefficients at B, see Eqs. (43) and (74)
C_{ij}	coefficients of constitutive equations, see Eq. (1)
$\Delta\psi$	see Eq. (42)
$\bar{C}_{12}$, $\bar{C}_{44}$	see Eqs. (60)
e	eccentricity (distance from shear center of shell wall to point of load application, positive outward)
H	horizontal (radial) force/unit length (see Fig. 5)
K	Gaussian curvature
k*	curvature of deformed shell
L	shell length (cylinders only)
M, $\bar{M}$	moment resultant, applied external moment
M_T	$M_{12} + M_{21}$
N	stress resultant
n	number of circumferential waves in buckling pattern
P	total axial load applied to shell, positive compression
p	normal pressure, positive internal

Q	shear load/unit length (see Fig. 5)
R	radius of curvature
r	horizontal radius from axis of rotation to middle surface
s	arc length measured from point A (see Fig. 5)
u	meridional displacement
u_H	horizontal (radial) displacement
u_V	veritical (axial) displacement
v	circumferential displacement
V	vertical (axial) force/unit length
$\bar{V}$	applied verticial force to shell, positive for tension
w	normal displacement, positive outward
$\bar{w}$	see Eq. (52)
β	meridional rotation (also winding angle for fiber-reinforced shells)
ϵ	middle surface strain
κ	change in curvature
ψ	stress function for prebuckling problem $\psi = rH$
φ	Airy-type stress function
ρ	loading parameter, e.g. p/p_{cr}
θ	circumferential coordinate
ξ	see Eqs. (50)

Subscripts and superscripts

$(\ )'$	differentiation with respect to arc length s
$(\ )^{\cdot}$	differentiation with respect to circumferential coordinate θ
$(\ )_x$	differentiation with respect to x
$(\ )_1$	pertains to meridional direction
$(\ )_2$	pertains to circumferential direction
$(\ )_{12}$	shear resultant, twisting moment, twisting change in curvature
$(\ )_o$, $(\ )_0$	prebuckling quantity
$(\ )_{cr}$	critical load
$(\ )_A$	at end A of the meridian (see Fig. 5)
$(\ )_B$	at end B of the meridian (see Fig. 5)

CONTENTS

Section 1

CONSTITUTIVE RELATIONS

In their most general form the relations between stresses and strains can be written as

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_T \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} \qquad (1)$$

The stiffness coefficients, C_{ij} , are determined here in terms of the elastic and geometric properties of some common types of shell wall design.

In the following discussion and in the numerical analysis the C_{ij} are assumed to be independent of the meridional arc length. There are cases (such as meridionally stiffened spherical domes or shells of variable thickness) for which the C_{ij} vary along a meridian. However, for the shells investigated in the numerical analysis presented here the C_{ij} are almost constant.

As in Ref. 1 the stiffness coefficients are obtained by use of the strain energy expression for the deformed shell. The strain energy, U, is expressed in terms of strains and changes of curvature, and the coefficients are found as follows: Let a subscript i following a comma indicate differentiation with respect to one of the strains or changes of curvatures such that for $i = 1, 2, 3, 4, 5, 6$ derivatives are taken with respect to ϵ_1, ϵ_2, ϵ_{12}, κ_1, κ_2, κ_{12}, respectively. Then

$$C_{ij} = U_{,ij} \tag{2}$$

The derivation of the constitutive equations will be shown in detail here only for one example, the fiber-reinforced shell.

1.1 Fiber-reinforced Shells

In general fiber-reinforced shells are composed of several layers, with the fibers unidirectional in each layer. The properties of each layer are determined by use of the equations given by Tsai in Ref. 2. Current fabrication techniques generally result in shells in which the layers appear pair-wise, one layer with the winding angle $+\beta$ and one with the angle $-\beta$ with respect to the meridian. Here the elastic properties for each pair of layers are first established and finally the elastic constants are derived for a shell wall composed of a number of double layers.

For each layer the following properties are required:

E_f = modulus of fibers

ν_f	=	Poisson's ratio for fibers
E_m	=	modulus of matrix
ν_m	=	Poisson's ratio for matrix
t	=	thickness of double layer
β	=	angle of wind
χ	=	matrix content (by volume)

Additional parameters are the two correction factors introduced by Tsai: the misalignment factor, k, and the contiguity factor, C. For structures that are wound with prestressed fibers it appears reasonable to assume $k = 1.0$ (no misalignment). The contiguity factor has to do with the stiffness perpendicular to the fiber direction ($C = 0$ for isolated filaments and $C = 1$ for contiguous filaments). The factor C is higher, of course, if the matrix content is low, but it is generally closer to 0 than to 1. Tsai obtains good agreement between theory and test with $C = 0.2$.

The moduli E_{11} and E_{22} in directions parallel and perpendicular to the fibers, the shear modulus G, and Poisson's ratio ν_{12} are given in Ref. 2 in terms of fiber and matrix properties (E_f, ν_f, E_m, ν_m). It can be shown that

$$\nu_{21} = \nu_{12} E_{22} / E_{11} \tag{3}$$

and

$$
\begin{aligned}
\epsilon_1 &= \epsilon'_x \cos^2\beta + \epsilon'_y \sin^2\beta + \epsilon'_{xy} \cos\beta \sin\beta \\
\epsilon_2 &= \epsilon'_x \sin^2\beta + \epsilon'_y \cos^2\beta - \epsilon'_{xy} \cos\beta \sin\beta \qquad (4) \\
\epsilon_{12} &= \epsilon'_{xy} (\cos^2\beta - \sin^2\beta) + 2(\epsilon'_y - \epsilon'_x) \cos\beta \sin\beta
\end{aligned}
$$

where ϵ'_x and ϵ'_y represent the strains in the meridional and circumferential directions, ϵ'_{xy} represents the corresponding shear strain and ϵ_1 , ϵ_2 , ϵ_{12} are strains with respect to a coordinate system which is rotated an angle β .

The strains at the middle surface of the double layer are ϵ_x , ϵ_y , ϵ_{xy} and the changes of curvatures are κ_x , κ_y , κ_{xy} . Hence

Layer 1	Layer 2
$\epsilon'_x = \epsilon_x + \frac{t}{4}\kappa_x$	$\epsilon'_x = \epsilon_x - \frac{t}{4}\kappa_x$
$\epsilon'_y = \epsilon_y + \frac{t}{4}\kappa_y$	$\epsilon'_y = \epsilon_y - \frac{t}{4}\kappa_y$
$\epsilon'_{xy} = \epsilon_{xy} + \frac{t}{2}\kappa_{xy}$	$\epsilon'_{xy} = \epsilon_{xy} - \frac{t}{2}\kappa_{xy}$

(5)

In Layer 1

$$\epsilon_1 = \left(\epsilon_x + \frac{t}{4}\kappa_x\right)\cos^2\beta + \left(\epsilon_y + \frac{t}{4}\kappa_y\right)\sin^2\beta + \left(\epsilon_{xy} + \frac{t}{4}\kappa_{xy}\right)\cos\beta\sin\beta$$

$$\epsilon_2 = \left(\epsilon_x + \frac{t}{4}\kappa_x\right)\sin^2\beta + \left(\epsilon_y + \frac{t}{4}\kappa_y\right)\cos^2\beta - \left(\epsilon_{xy} + \frac{t}{4}\kappa_{xy}\right)\cos\beta\sin\beta \qquad (6)$$

$$\epsilon_{12} = \left(\epsilon_{xy} + \frac{t}{4}\kappa_{xy}\right)(\cos^2\beta - \sin^2\beta) + 2\left(\epsilon_y + \frac{t}{4}\kappa_y - \epsilon_x - \frac{t}{4}\kappa_x\right)\cos\beta\sin\beta$$

In Layer 2

$$\epsilon_1 = \left(\epsilon_x - \frac{t}{4}\kappa_x\right)\cos^2\beta + \left(\epsilon_y - \frac{t}{4}\kappa_y\right)\sin^2\beta - \left(\epsilon_{xy} - \frac{t}{4}\kappa_{xy}\right)\cos\beta\sin\beta$$

$$\epsilon_2 = \left(\epsilon_x - \frac{t}{4}\kappa_x\right)\sin^2\beta + \left(\epsilon_y - \frac{t}{4}\kappa_y\right)\cos^2\beta + \left(\epsilon_{xy} - \frac{t}{4}\kappa_{xy}\right)\cos\beta\sin\beta \qquad (7)$$

$$\epsilon_{12} = \left(\epsilon_{xy} - \frac{t}{4}\kappa_{xy}\right)(\cos^2\beta - \sin^2\beta) - 2\left(\epsilon_y - \frac{t}{4}\kappa_y - \epsilon_x + \frac{t}{4}\kappa_x\right)\cos\beta\sin\beta$$

The following expressions for the changes of curvature apply in the reference systems with coordinates parallel or perpendicular to the fiber directions

$$\kappa_1 = \kappa_x\cos^2\beta + \kappa_y\sin^2\beta \pm 2\kappa_{xy}\cos\beta\sin\beta$$

$$\kappa_2 = \kappa_x\sin^2\beta + \kappa_y\cos^2\beta \mp 2\kappa_{xy}\cos\beta\sin\beta \qquad (8)$$

$$\kappa_{12} = \kappa_{xy}(\cos^2\beta - \sin^2\beta) \pm (\kappa_x - \kappa_y)\cos\beta\sin\beta$$

where, when there are double signs, the upper applies to Layer 1 and the lower to Layer 2.

The strain energy density in the double layer is

$$U = \left\{ \frac{t}{2} \left[\frac{\epsilon_1^2 E_{11}}{1 - \nu_{12}\nu_{21}} + \frac{\epsilon_2^2 E_{22}}{1 - \nu_{12}\nu_{21}} + \frac{2\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} \epsilon_1 \epsilon_2 + G\gamma_{12}^2 \right] \right.$$
$$\left. + \frac{t^3}{8 \cdot 12} \left[\frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \kappa_1^2 + \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \kappa_2^2 + \frac{2\nu_{12}E_{21}}{1 - \nu_{12}\nu_{21}} \kappa_1 \kappa_2 + 4G\kappa_{12}^2 \right] \right\} \qquad (9)$$

Substitution of Eqs.(6),(7), and(8) into Eq.(9) and subsequent applications of Eq.(2) gives the stiffness coefficients for one (double) layer. The matrix $[C_{ij}]$ is diagonally symmetrical. Thus there are 21 independent constants. The non-zero C_{ij}'s are

$$C_{11} = \frac{t}{1 - \nu_{12}\nu_{21}} \left[E_{11} \cos^4 \beta + E_{22} \sin^4 \beta + \{2\nu_{12}E_{22} + 4G(1 - \nu_{12}\nu_{21})\} \cos^2 \beta \sin^2 \beta \right]$$

$$C_{12} = \frac{t}{1 - \nu_{12}\nu_{21}} \left[\left\{ E_{11} + E_{22} - 4G(1 - \nu_{12}\nu_{21}) \right\} \sin^2 \beta \cos^2 \beta + \nu_{12}E_{22}(\cos^4 \beta + \sin^4 \beta) \right]$$

$$C_{16} = \frac{t^2}{4(1 - \nu_{12}\nu_{21})} \sin\beta \cos\beta \left[E_{11} \cos^2 \beta - E_{22} \sin^2 \beta + \nu_{12}E_{22} (\sin^2\beta - \cos^2\beta) \right.$$
$$\left. - 2G(1 - \nu_{12}\nu_{21})(\cos^2 \beta - \sin^2 \beta) \right]$$

$$C_{22} = \frac{t}{1 - \nu_{12}\nu_{21}} \left[E_{11} \sin^4 \beta + E_{22} \cos^4 \beta + \{2\nu_{12}E_{22} + 4G(1 - \nu_{12}\nu_{21})\} \sin^2\beta \cos^2\beta\right]$$

$$C_{26} = \frac{t^2}{4(1 - \nu_{12}\nu_{21})} \sin \beta \cos \beta \left[E_{11} \sin^2 \beta - E_{22} \cos^2 \beta + \nu_{12}E_{22} (\cos^2\beta - \sin^2\beta)\right.$$

$$\left. + 2G(1 - \nu_{12}\nu_{21})(\cos^2 \beta - \sin^2 \beta)\right]$$

$$C_{33} = \frac{t}{(1 - \nu_{12}\nu_{21})} \left[(E_{11} + E_{22} - 2\nu_{12}E_{22}) \cos^2 \beta \sin^2 \beta\right.$$

$$\left. + G(1 - \nu_{12}\nu_{21})(\cos^2 \beta - \sin^2 \beta)^2\right]$$

$$C_{34} = \frac{t^2}{4(1 - \nu_{12}\nu_{21})} \cos \beta \sin \beta \left[E_{11} \cos^2 \beta - E_{22} \sin^2 \beta + \nu_{12}E_{22} (\sin^2\beta \;\; \cos^2\beta)\right.$$

$$\left. - 2G(1 - \nu_{12}\nu_{21})(\cos^2 \beta - \sin^2 \beta)\right]$$

$$C_{35} = \frac{t^2}{4(1 - \nu_{12}\nu_{21})} \cos \beta \sin \beta \left[E_{11} \sin^2 \beta - E_{22} \cos^2 \beta \;\; \nu_{12}E_{22} \;\; (\cos^2\beta \;\; \sin^2\beta)\right.$$

$$\left. + 2G(1 - \nu_{12}\nu_{21})(\cos^2 \beta - \sin^2 \beta)\right]$$

$$C_{44} = \frac{t^3}{12(1 - \nu_{12}\nu_{21})} \left\{E_{11} \cos^4 \beta + E_{22} \sin^4 \beta + (2\nu_{12}E_{22} + 4G) \sin^2 \beta \cos^2 \beta\right\}$$

$$C_{45} = \frac{t^3}{12(1 - \nu_{12}\nu_{21})} \left\{\sin^2 \beta \cos^2 \beta (E_{11} + E_{22} - 4G) + \nu_{12}E_{22} (\cos^4 \beta + \sin^4 \beta)\right\}$$

$$C_{55} = \frac{t^3}{12(1 - \nu_{12}\nu_{21})} \left[E_{11} \sin^4 \beta + E_{22} \cos^4 \beta + (\nu_{12}E_{22} + 4G) \sin^2 \beta \cos^2 \beta\right]$$

$$C_{66} = \frac{t^3 \cos^2 \beta \sin^2 \beta}{3 (1 - \nu_{12}\nu_{21})} [E_{11} + E_{22} - \nu_{12}E_{22}] + \frac{t^3}{3} \cdot G (\cos^2 \beta - \sin^2 \beta)^2 \qquad (10)$$

In general a fiber-reinforced shell is composed of a number of double layers. The elastic properties for each of these are obtained from Eqs. (10). Hence for the i^{th} layer

$$
\begin{aligned}
N_1^i &= C_{11}^i \epsilon_1^i + C_{12}^i \epsilon_2^i + C_{16}^i \kappa_{12} \\
N_2^i &= C_{12}^i \epsilon_1^i + C_{22}^i \epsilon_2^i + C_{26}^i \kappa_{12} \\
N_{12}^i &= C_{33}^i \epsilon_{12}^i + C_{34}^i \kappa_1 + C_{35}^i \kappa_2 \\
M_1^i &= C_{34}^i \epsilon_{12}^i + C_{44}^i \kappa_1 + C_{45}^i \kappa_2 \\
M_2^i &= C_{35}^i \epsilon_{12}^i + C_{45}^i \kappa_1 + C_{55}^i \kappa_2 \\
M_T^i &= C_{16}^i \epsilon_1^i + C_{46}^i \kappa_2 + C_{66}^i \kappa_{12}
\end{aligned}
$$

Here ϵ_1^i , ϵ_2^i , and ϵ_{12}^i are strains at the midplane of the i^{th} layer. The strain density in one layer can be written in the form

$$
\begin{aligned}
U^i = \frac{1}{2} \Big\{ & C_{11}^i \left(\epsilon_1^i\right)^2 + 2\, C_{12}^i \epsilon_1^i \epsilon_2^i + 2\, C_{16}^i \epsilon_1^i \kappa_{12} + C_{22}^i \left(\epsilon_2^i\right)^2 \\
& + 2\, C_{26}^i \epsilon_2^i \kappa_{12} + C_{33}^i \left(\epsilon_{12}^i\right)^2 + 2\, C_{34}^i \epsilon_{12}^i \kappa_1 + 2\, C_{35}^i \epsilon_{12}^i \kappa_2 \\
& + C_{44}^i \left(\kappa_1\right)^2 + 2\, C_{45}^i \kappa_1 \kappa_2 + C_{55}^i \left(\kappa_2\right)^2 + C_{66}^i \left(\kappa_{12}\right)^2 \Big\}
\end{aligned}
\tag{11}
$$

If ϵ_1 , ϵ_2 , ϵ_{12} are strains at any given reference surface the strains in the individual layers are given by

$$
\begin{aligned}
\epsilon_1^i &= \epsilon_1 - h_i \kappa_1 \\
\epsilon_2^i &= \epsilon_2 - h_i \kappa_2 \\
\epsilon_{12}^i &= \epsilon_{12} - 2\, h_i \kappa_2
\end{aligned}
\tag{12}
$$

where h_i is defined in Fig. 1

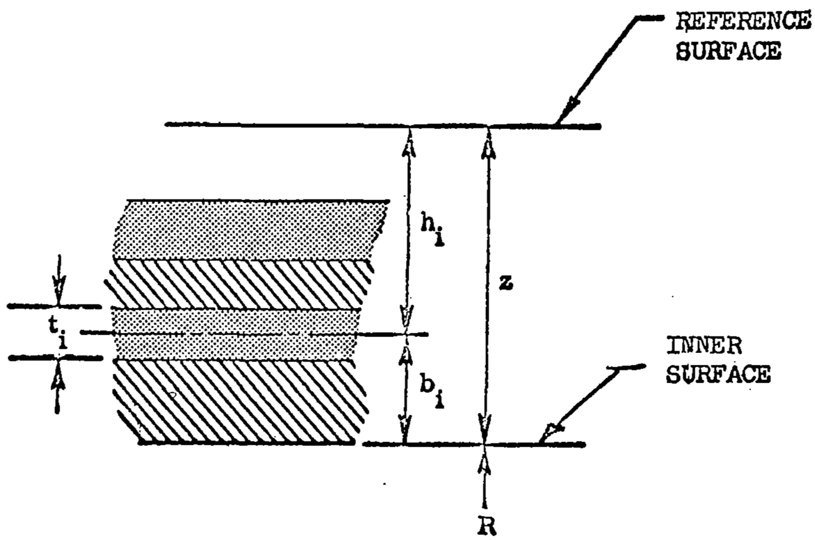

Fig. 1 Layered Shell

If the total number of layers is I , substitution of Eqs. (12) into Eqs. (11) and subsequent application of Eqs. (2) yields

$$C_{11} = \sum C^i_{11} \qquad C_{12} = \sum C^i_{12}$$

$$C_{13} = 0 \qquad C_{14} = \sum - h_i\, C^i_{12}$$

$$C_{15} = \sum - h_i\, C^i_{12} \qquad C_{16} = \sum C^i_{16}$$

$$C_{22} = \sum C^i_{22} \qquad C_{23} = 0$$

$$C_{24} = -\sum h_i C^i_{12} = C_{15} \qquad C_{25} = -\sum h_i C^i_{22}$$

$$C_{26} = \sum C^i_{26} \qquad C_{33} = \sum C^i_{33}$$

$$C_{34} = \sum C^i_{34} \qquad C_{35} = \sum C^i_{35}$$

$$C_{36} = -\sum 2 h_i C^i_{33} \qquad C_{44} = \sum (h_i^2 C^i_{11} + C^i_{44})$$

$$C_{45} = \sum (h_i^2 C^i_{12} + C^i_{45}) \qquad C_{46} = -\sum h_i (C^i_{26} + C^i_{45})$$

$$C_{55} = \sum (h_i^2 C^i_{22} + C^i_{55}) \qquad C_{56} = -\sum h_i (C^i_{26} + C^i_{35})$$

$$C_{66} = \sum (4h_i^2 C^i_{33} + C^i_{66}) \tag{13}$$

where

$$\sum = \sum_{i=1}^{I}$$

For this example as well as for all other types of shell walls considered here the reference surface is chosen such that

$$C_{36} = 0$$

That is

$$z = \left(\sum_{i=1}^{I} b_i C^i_{33} \right) / \left(\sum_{i=1}^{I} C^i_{33} \right)$$

The present stability analysis is restricted to the case in which axially symmetrical loads on the shell lead to axially symmetrical prebuckling deformations. Consequently the analysis is applicable only if

$$C_{13} = C_{16} = C_{23} = C_{26} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \qquad (14)$$

According to Eqs. (13) only C_{13} and C_{23} are zero, but it appears that in most cases the remaining constants in Eqs.(14)are negligible.

For the present analysis it will thus be assumed that

$$\begin{aligned}
N_1 &= C_{11}\,\epsilon_1 + C_{12}\,\epsilon_2 + C_{14}\,\kappa_1 + C_{15}\,\kappa_2 \\
N_2 &= C_{12}\,\epsilon_1 + C_{22}\,\epsilon_2 + C_{24}\,\kappa_1 + C_{25}\,\kappa_2 \\
N_{12} &= C_{33}\,\epsilon_{12} \\
M_1 &= C_{14}\,\epsilon_1 + C_{24}\,\epsilon_2 + C_{44}\,\kappa_1 + C_{45}\,\kappa_2 \\
M_2 &= C_{15}\,\epsilon_1 + C_{25}\,\epsilon_2 + C_{45}\,\kappa_1 + C_{55}\,\kappa_2 \\
M_T &= C_{66}\,\kappa_{12}
\end{aligned} \qquad (15)$$

where the constants are given by Eqs. (13)

Stiffness coefficients for a number of shells of different wall constructions have been derived in a similar way and the results are presented below.

1.2 Shells Stiffened by Rings and Stringers (isotropic skin)

For the determination of the coefficients of Eqs.(15), the following properties are required:

For skin	E	Young's modulus
	ν	Poisson's ratio
	t	thickness

For stiffeners

d_μ	Spacing
A_μ	Cross-sectional area
I_μ	Moment of inertia of cross-section around its neutral axis parallel to skin
J_μ	Torsional constant
X_μ	Distance from neutral surface to skin midplane
E_μ	Young's modulus
G_μ	Shear modulus
	Position (outside or inside)

These data are for stringers if the subscript μ equals one and for rings if it equals two. For the special case of integral rectangular stiffeners it is sufficient to define the spacing and

b_μ	Stiffener width
h_μ	Stiffener height

For this case

$$\begin{aligned}
A_\mu &= b_\mu h_\mu \\
I_\mu &= b_\mu h_\mu^3/12 \\
J_\mu &= \frac{1}{3} \text{ Min } (b_\mu h_\mu^3 , b_\mu^3 h_\mu) \\
X_\mu &= (b_\mu + t)/2 \\
E_\mu &= E \\
G_\mu &= E/[2(1 + \nu)]
\end{aligned} \tag{16}$$

For brevity the following parameters are introduced

$$\mu_1 = (1 - \nu^2)\, A_1/(t\, d_1)$$

$$\mu_2 = (1 - \nu^2)\, A_2/(t\, d_2)$$

$$\eta_1 = (I_1 + A_1\, X_1^2)\, (1 - \nu^2)/(d_1\, t)$$

$$\eta_2 = (I_2 + A_2\, X_2^2)\, (1 - \nu^2)/(d_2\, t)$$

$$\eta_{t1} = (1 - \nu^2)\, G_1\, J_1/(d_1\, t\, E)$$

$$\eta_{t2} = (1 - \nu^2)\, G_2\, J_2/(d_2\, t\, E) \qquad (17)$$

$$C = E\, t/(1 - \nu^2)$$

With the skin middle surface as reference surface the stiffness coefficients are

$$C_{11} = C\, (1 + \mu_1)$$

$$C_{12} = \mu\, C$$

$$C_{14} = X_1\, \mu_1\, C$$

$$C_{15} = 0$$

$$C_{22} = C\, (1 + \mu_2)$$

$$C_{24} = 0$$

$$C_{25} = X_2\, \mu_2\, C$$

$$C_{33} = 0.5\, (1 - \nu)\, C$$

$$C_{44} = C\, [(1 + \eta_1) + t^2/12]$$

$$C_{45} = \nu\, C\, t^2/12$$

$$C_{55} = C\, [(1 + \eta_2) + t^2/12] \qquad (18)$$

$$C_{66} = C\, [\frac{t^2}{6}\, (1 - \nu) + \eta_{t1} + \eta_{t2}]$$

These constants are identical to those in Ref. 3 .

1.3 Shells with Skew Stiffeners

It is assumed that the skin is isotropic,and that the stiffeners are rectangular monolithic

The following information is required

For skin	E	Young's modulus
	μ	Poisson's ratio
	t	thickness
For stiffener	θ	angle between stiffener and generator
	d	spacing (see Fig. 2)
	b	width
	h	height
	Position (outside or inside)	

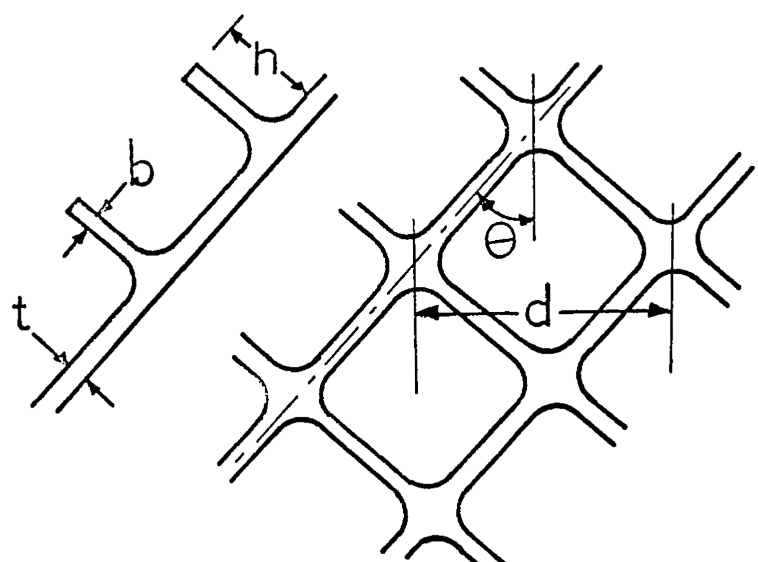

Fig. 2 Skew Stiffeners

The following notations are introduced:

$$
\begin{aligned}
C &= E\, t/(1 - \nu^2) \\
\mu &= 2\, b\, h\, (1 - \nu^2)/(d\, t) \\
\gamma^2 &= t^2/12 \\
\eta &= b\, h^3\, (1 - \nu^2)/(6\, d\, t) \\
\eta_T &= 2\, h\, b^3\, (1 - \nu)/(3\, d\, t) \\
\rho &= 0.5\, (h + t) \\
\rho_1 &= \rho/[1 + 2\mu \cos\theta \sin^2\theta/(1 - \nu)] \\
\rho_2 &= \rho/[1 + (1 - \nu)/(2\mu \cos\theta \sin^2\theta)]
\end{aligned}
\qquad (19)
$$

where ρ_2 is the distance from the skin's midsurface to the reference surface. For the case of internal stiffeners ρ is negative.

The stiffness coefficients are

$$
\begin{aligned}
C_{11} &= C\, (1 + \mu \cos^3\theta) \\
C_{12} &= C\, (\nu + \mu \cos\theta \sin^2\theta) \\
C_{14} &= C\, (\rho_2 - \mu\, \rho_1 \cos^3\theta) \\
C_{15} &= C\, (\nu\, \rho_2 - \mu\, \rho_1 \cos\theta \sin^2\theta) \\
C_{22} &= C\, (1 + \mu \sin^4\theta/\cos\theta) \\
C_{24} &= C\, (\nu\, \rho_2 - \mu\, \rho_1 \cos\theta \sin^2\theta) \\
C_{25} &= C\, (\rho_2 - \mu\, \rho_1 \sin^4\theta/\cos\theta) \\
C_{33} &= 0.5\, (1 - \nu) + \mu \cos\theta \sin^2\theta \\
C_{44} &= C[\gamma^2 + \rho_2^2 + (\mu\, \rho_1^2 + \eta) \cos^3\theta + \eta_T \sin^2\theta \cos\theta]
\end{aligned}
$$

$$C_{45} = C\,[\nu\,\gamma^2 + \nu\,\rho_2^2 + (\mu\,\rho_1^2 + \eta)\cos\theta\,\sin^2\theta - \eta_T\,\sin^2\theta\,\cos\theta]$$

$$C_{55} = C\,[\gamma^2 + \rho_2^2 + (\mu\,\rho_1^2 + \eta)\,\sin^4\theta/\cos\theta + \eta_T\sin^2\theta\,\cos\theta] \tag{20}$$

$$C_{66} = C\,[2(1 - \nu)\,\gamma^2 + 2\,(1 - \nu)\,\rho_2^2 + 4\,(\mu\,\rho_1^2 + \eta)\cos\theta\,\sin^2\theta + \eta_T\,\cos^2 2\theta/\cos\theta]$$

1.4 Orthotropic Skin with Stiffeners

For the shell without stiffeners the following relations apply:

$$
\begin{aligned}
N_1 &= \bar{C}_{11}\,\epsilon_1 + \bar{C}_{12}\,\epsilon_2 + \bar{C}_{14}\,\kappa_1 + \bar{C}_{15}\,\kappa_2 \\
N_2 &= \bar{C}_{12}\,\epsilon_1 + \bar{C}_{22}\,\epsilon_2 + \bar{C}_{24}\,\kappa_1 + \bar{C}_{25}\,\kappa_2 \\
N_{12} &= \bar{C}_{33}\epsilon_{12} \\
M_1 &= \bar{C}_{14}\epsilon_1 + \bar{C}_{24}\,\epsilon_2 + \bar{C}_{44}\,\kappa_1 + \bar{C}_{45}\,\kappa_2 \\
M_2 &= \bar{C}_{15}\epsilon_1 + \bar{C}_{25}\,\epsilon_2 + \bar{C}_{45}\,\kappa_1 + \bar{C}_{55}\,\kappa_2 \\
M_T &= \bar{C}_{66}\,\kappa_{12}
\end{aligned} \tag{21}
$$

The most general case allowed then is a "skin" which is composed of a number of orthotropic layers. The $\bar{C}_{ij}$'s could for instance be obtained through application of the Eqs.(13)to a fiberglass reinforced shell

In addition the following information is required:

For skin t = total thickness

z = distance from inner surface to shear center

For stiffeners

$$\left.\begin{array}{l} d_\mu \\ A_\mu \\ I_\mu \\ J_\mu \\ E_\mu \\ G_\mu \end{array}\right\}$$ defined under "Shells Stiffened by Rings and Stringers"

$\bar{X}_\mu$ distance from the neutral axis of stiffener to the closest skin surface.

The subscript μ again is one for stringers (meridional stiffeners) and two for rings

Set

$$X_\mu = \begin{cases} \bar{X}_\mu + t - z & \text{for inside stiffeners} \\ -\bar{X}_\mu - z & \text{for outside stiffeners} \end{cases} \tag{22}$$

The following stiffness constants are obtained for the skin and stiffener combination

$$C_{11} = \bar{C}_{11} + E_1 A_1/d_1$$

$$C_{12} = \bar{C}_{12}$$

$$C_{14} = \bar{C}_{14} + X_1 E_1 A_1/d_1$$

$$C_{15} = \bar{C}_{15}$$

$$C_{22} = \bar{C}_{22} + E_2 A_2/d_2$$

$$
\begin{aligned}
C_{24} &= \bar{C}_{24} \\
C_{25} &= \bar{C}_{25} + X_2\, E_2\, A_2/d_2 \\
C_{33} &= \bar{C}_{33} \\
C_{44} &= \bar{C}_{44} + E_1\, (I_1 + X_1^2\, A_1)/d_1 \\
C_{45} &= \bar{C}_{45} \\
C_{55} &= \bar{C}_{55} + E_2\, (I_2 + X_2^2\, A_2)/d_2 \\
C_{66} &= \bar{C}_{66} + G_1\, J_1/d_1 + G_2\, J_2/d_2
\end{aligned} \qquad (23)
$$

These equations may be used also if stringers and rings are added to the configurations discussed in the sequel.

1.5 Layered Shells

For layered shells the stiffness coefficients can be established on basis of the following information:

	I	Number of layers
For each layer	$(E_x)_i$	Young's Modulus in axial direction
	$(E_y)_i$	Young's Modulus in circumferential direction
	$(\nu_{xy})_i$	Poisson's Ratio
	G_i	Shear Modulus
	t_i	thickness

The distance from the inner surface of the shell to the middle surface of the i^{th} layer is

$$b_i = \frac{t_i}{2} + \sum_{j=1}^{i-1} t_i \qquad (24)$$

The shear center is located at a distance z outside of the inner surface where

$$z = \sum_{i=1}^{I} (b_i\ G_i\ t_i) / \sum_{i=1}^{I} G_i\ t_i \qquad (25)$$

With

$$\begin{aligned}
(\nu_{yx})_i &= (\nu_{xy})_i\ (E_x)_i/(E_y)_i \\
\nu_i^2 &= (\nu_{xy})_i\ (\nu_{yx})_i
\end{aligned} \qquad (26)$$

and

$$h_i = z - b_i$$

the stiffness coefficients for layered shells are

$$\bar{C}_{11} = \sum_{i=1}^{I} (E_x)_i\ t_i/(1 - \nu_i^2)$$

$$\bar{C}_{12} = \sum_{i=1}^{I} (\nu_{xy})_i\ (E_x)_i\ t_i/(1 - \nu_i^2)$$

$$\bar{C}_{14} = \sum_{i=1}^{I} h_i\ (E_x)_i\ t_i/(1 - \nu_i^2)$$

$$\bar{C}_{15} = \sum_{i=1}^{I} h_i\ (\nu_{xy})_i\ (E_x)_i\ t_i/(1 - \nu_i^2)$$

$$\bar{C}_{22} = \sum_{i=1}^{I} (E_y)_i\ t_i/(1 - \nu_i^2)$$

$$\bar{C}_{24} = \bar{C}_{15}$$

$$\bar{C}_{25} = \sum_{i=1}^{I} h_i \, (E_y)_i \, t_i/(1 - \nu_i^2)$$

$$\bar{C}_{33} = \sum_{i=1}^{I} G_i \, t_i$$

$$\bar{C}_{44} = \sum_{i=1}^{I} (t_i^2/12 + h_i^2) \, (E_x)_i \, t_i/(1 - \nu_i^2)$$

$$\bar{C}_{45} = \sum_{i=1}^{I} (\nu_{xy})_i \, (t_i^2/12 + h_i^2) \, (E_x)_i \, t_i/(1 - \nu_i^2)$$

$$\bar{C}_{55} = \sum_{i=1}^{I} (t_i^2/12 + h_i^2) \, (E_y)_i \, t_i/(1 - \nu_i^2)$$

$$\bar{C}_{66} = \sum_{i=1}^{I} 4 \, (t_i^2/12 + h_i^2) \, G_i \, t_i \qquad (27)$$

1.6 Corrugated Skin with Rihg-Stiffeners

The present analysis is applicable only for shells of revolution. Consequently it does not apply to shells with stiffeners in the meridional direction unless the stiffeners are sufficiently closely spaced. If the stringer spacing is small in comparison to the circumferential wave length of the buckling pattern the properties of the stiffener can be "smeared out" in an obvious way. For the corrugated skin a similar simplification is readily available for determination of some of the stiffness coefficients. However, for other parameters such as the torsional stiffness this is not the case and certain loosely-founded assumptions are made in the development. By variation of the questionable parameters in the numerical analysis it is revealed that the critical axial load is only moderately effected.

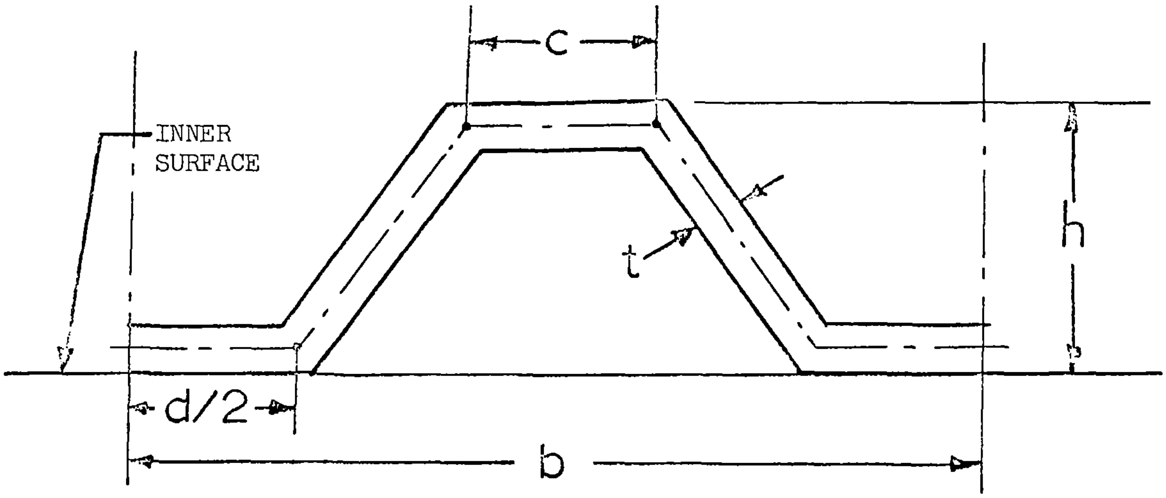

Fig. 3 Corrugated Skin

The following data are required for the analysis

E	Young's modulus
ν	Poisson's ratio
h, t, b, c, d	dimensions in Fig. 3

The following notations are also used:

$$\begin{aligned} \beta &= (b - c - d)/[2(h - t)] \\ \alpha &= (1 + \beta^2)^{1/2} \\ f &= c - t(\alpha + \beta) \\ g &= d - t(\alpha + \beta) \end{aligned} \tag{28}$$

The area per unit width

$$t_x = \frac{2}{b} \left[\frac{f + g}{2} + h\,\alpha + t\,\beta \right] \tag{29}$$

Distance from inner surface to neutral surface

$$z = t \left[\frac{t}{2} (g - f) + h\,(f + \beta t + \alpha h) \right] / (b t_x) \tag{30}$$

Moment of inertia per unit width

$$\begin{aligned} I_x = \frac{t}{b} \Big[\frac{1}{6} \Big\{ \alpha h^3 + t^2 \Big(\frac{g + f}{2} + \beta t \Big) \Big\} &+ 2\alpha h \left(\frac{h}{2} - z\right)^2 + g\left(z - \frac{t}{2}\right)^2 \\ + f\left(h - z - \frac{t}{2}\right)^2 &+ \beta t \left\{ \left(z - \frac{2}{3} t\right)^2 + \left(h - z - \frac{2}{3} t\right)^2 \right\} \Big] \end{aligned} \tag{31}$$

The bending stiffness in the circumferential direction is $E\,\bar{I}_y$ where

$$\bar{I}_y = \frac{t^3}{12(1 - \nu^2)} \frac{b}{L} \tag{32}$$

and L is the developed width of the corrugation.

$$L = 2\,\alpha\,(h - t) + c + d \tag{33}$$

Under the assumption that the shear force in the plane of the sheet is constant the shear stiffness is $G\,\bar{t}$ where

$$\bar{t} = \frac{b}{L} \tag{34}$$

It is assumed that the torsional stiffness can be represented by that of a narrow circumferential strip. The torsional stiffness is G $\bar{J}$ where

$$\bar{J} = \frac{t^3}{3} / \left[\beta/\alpha + \frac{c + d}{b} (1 - \beta/\alpha)\right] \tag{35}$$

The coefficients in the constitutive relations for the skin are

$$\begin{aligned}
\bar{C}_{11} &= E\, t_x \\
\bar{C}_{12} &= \bar{C}_{14} = \bar{C}_{15} = 0 \\
\bar{C}_{22} &= \bar{C}_{24} = \bar{C}_{25} = 0 \\
\bar{C}_{33} &= G\, \bar{t} \\
\bar{C}_{44} &= E\, I_x \\
\bar{C}_{45} &= 0 \\
\bar{C}_{55} &= E\, \bar{I}_y \\
\bar{C}_{66} &= G\, \bar{J}
\end{aligned} \tag{36}$$

The coefficients for the ring stiffened shells are found by use of equations given above (orthotropic skin with stiffeners).

1.7 Semi-sandwich

A shell wall composed of a corrugated sheet attached to a smooth skin as shown in Fig. 4 is here referred to as a semi-sandwich. The shell may or may not be reinforced by rings.

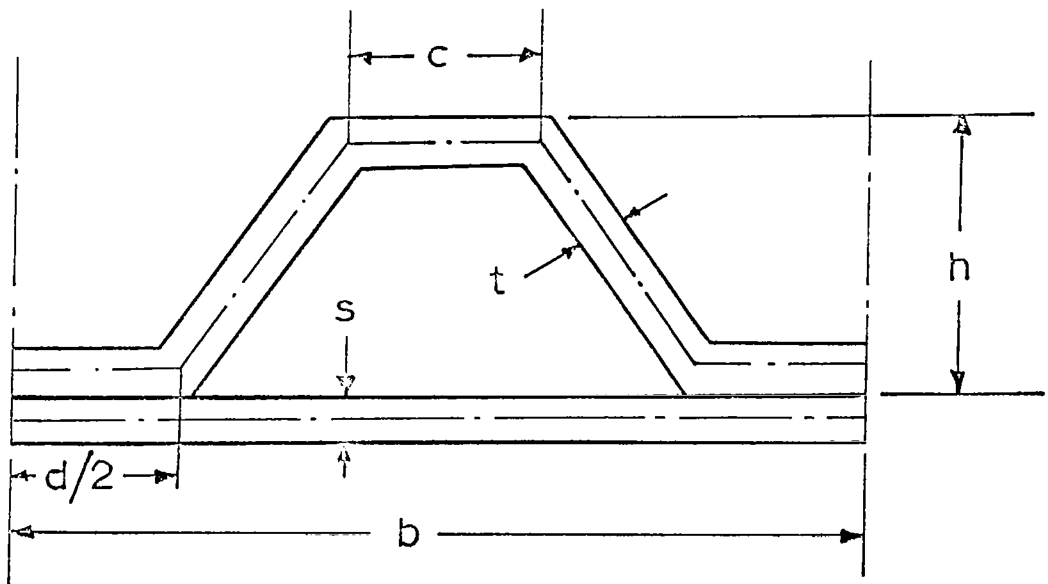

Fig. 4 Semi-sandwich

In addition to the problems discussed above for the corrugated skin, the determination of the torsional stiffness of the shell wall is questionable. The combination of corrugation and skin provides a closed section that can carry efficiently a twisting moment M_T . However, this is true only if the moment is independent of the axial coordinate. If the moment varies, local bending is introduced and the stiffness is greatly reduced. Obviously the torsional stiffness of the closed section depends on the pattern of deformation of the shell and a rigorous solution cannot be obtained by use of an equivalent orthotropic shell. In the computer program based on the present analysis a reduction factor φ to the torsional stiffness is an input parameter. The data required for the determination of the stiffness coefficients are

E, ν, h, t, b, c, d	Properties of corrugated skin as defined above (Corrugated Skin with Ring Stiffeners).
E_s	Young's modulus of plane skin
ν_s	Poisson's ratio of plane skin
s	thickness of plane skin

Position of corrugation (outside or inside)

φ	correction factor

The following notations are also used

t_x, $\bar{t}$, I_x, $\bar{I}_y$, $\bar{J}$ as defined above (Corrugated Skin with Ring Stiffeners)

$$G = E/[2(1 + \nu)]$$

$$G_s = E_s/[2(1 + \nu_s)]$$

$$t^* = G\bar{t}(h + s)/[2\ G_s\ (s + \bar{t})]$$

$$t' = t[\tfrac{t}{2}(g - f) + h(f + \beta t + \alpha h)]/(bt_x)$$

$$e_1 = \begin{cases} (t' - t^* + s/2) & \text{for outside corrugation} \\ -(t' - t^* + s/2) & \text{for inside corrugation} \end{cases}$$

$$e_2 = \begin{cases} t^* & \text{for outside corrugation} \\ -t^* & \text{for inside corrugation} \end{cases}$$

$$C = E_s\ s/(1 - \nu_s^2)$$

$$D = E_s\ s^3/[12(1 - \nu_s^2)]$$

$$a = 0.5\ (h - t/2)\ (c + b - d)$$

The coefficients in the constitutive relations for the semi-sandwich (without rings) are

$$C_{11} = Et_x + C$$

$$C_{12} = \nu_s C$$

$$C_{14} = e_1 Et_x - e_2 C$$

$$C_{15} = -\nu_s e_2 C$$

$$C_{22} = C$$

$$C_{24} = -\nu_s e_2 C$$

$$C_{25} = -e_2 C$$

$$C_{33} = sG_s + \bar{t}G$$

$$C_{44} = D + EI_x + Et_x e_1^2 + Ce_2^2$$

$$C_{45} = \nu_s\ D$$

$$C_{55} = D + E\bar{I}_y + Ce_2^2$$

$$C_{66} = GJ + G_s s^3/3 + 2\varphi a^2/[b\{(b - d)/(sG_s) + (C + \alpha h - \alpha t)/(Gt)\}]$$

Section 2

PREBUCKLING ANALYSIS

2.1 General Shells of Revolution

The axisymmetric prebuckling equilibrium state of the shell is governed by two nonlinear, nonhomogeneous, ordinary differential equations of second order: an equation of equilibrium of forces normal to the undeformed meridian and an equation of compatibility of strains. The equations, applicable to shells of general wall construction are similar to those derived by Reissner (Ref. 4, Eqs. III and IV) for isotropic shells. They are valid for "small, finite rotations", that is the square of the meridional rotation is neglected compared to unity. These equations are derived in Ref. 5 .

Since the equations are solved by the method of finite-differences, their coefficients must be evaluated at each of the meridional stations in the finite-difference mesh. In the computer program the finite-difference equations are arranged such that the matrix of coefficients is strongly banded about the main diagonal: the compatibility and equilibrium equations alternate. Hence, the differential equations have the following form:

- Compatibility: Odd Equation Numbers: M = 2I + 1, I = 1, 2, 3, ...K + 1 (K is the number of intervals in the finite difference mesh)

$$
\begin{aligned}
&C(M,1)\psi'' + C(M,2)\psi' + C(M,3)\psi + C(M,4)\beta'' + [CNONL(M,1) + CNONL(M,2)*\beta \\
&+ CNONL(M,3)*\Delta\psi + CNONL(M,4)*p]\beta' + [CNONL(M,5) + CNONL(M,6)*\beta \\
&+ CNONL(M,7)*\Delta\psi + CNONL(M,8)*rV + CNONL(M,9)*p]\beta \;=\; FO(M)*rV + F1(M)*p
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
C(M,1) &= A_{22}r \\
C(M,2) &= A'_{22}r + A_{22}r' \\
C(M,3) &= A'_{12}r' - A_{12}r/R_1R_2 - A_{11}r'^2/r \\
C(M,4) &= A_{23}r \\
CNONL(M,1) &= A'_{23}r + r'(A_{23} + A_{24} - A_{13}) \\
CNONL(M,2) &= (A_{24} - A_{13})r/R_2 \\
CNONL(M,3) &= A_{12}r/R_2 \\
CNONL(M,4) &= -A_{22}r'r^2 \\
CNONL(M,5) &= A'_{24}r' - A_{24}r/R_1R_2 - A_{14}r'^2/r - r/R_2 \\
CNONL(M,6) &= A'_{24}r/2R_2 + A_{24}r'/2R_1 - 3A_{14}r'/2R_2 + r'/2 \\
CNONL(M,7) &= A'_{12}r/R_2 + A_{12}r'/R_1 - 2A_{11}r'/R_2 \\
CNONL(M,8) &= A_{12}r'^2R_2/rR_1 - A_{11}/r + A_{12}r/R_1R_2 \\
CNONL(M,9) &= A_{22}r(r^2/R_1R_2 - 2r'^2) - A'_{22}r^2r' \\
FO(M) &= A_{11}r'/R_2 - A_{12}r'/R_1 - A'_{12}r/R_2 \\
F1(M) &= -r^2(A_{22}r'/R_1 + 2A_{22}r'/R_2 + A'_{22}r/R_2)
\end{aligned}
\tag{38}
$$

- <u>Equilibrium</u>: Even Equation Numbers: $M = 2I + 2$, $I = 1, 2, 3, \ldots K + 1$

$$\begin{aligned}
&\mathrm{C(M,1)}\,\psi'' + [\mathrm{CNONL(M,1)} + \mathrm{CNONL(M,2)}*\beta]\,\psi' + \mathrm{C(M,3)}\,\psi + \mathrm{C(M,4)}\beta'' \\
&+ [\mathrm{CNONL(M,3)} + \mathrm{CNONL(M,4)}*\Delta\psi + \mathrm{CNONL(M,5)}*p]\beta' + [\mathrm{CNONL(M,6)} \\
&+ \mathrm{CNONL(M,7)}*\beta + \mathrm{CNONL(M,8)}*\Delta\psi + \mathrm{CNONL(M,9)}*rV + \mathrm{CNONL(M,10)}*p]\beta \\
&\qquad = \mathrm{F0(M)}*rV + \mathrm{F1(M)}*p
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\mathrm{C(M,1)} &= -A_{23}r \\
\mathrm{CNONL(M,1)} &= -A'_{23}r + r'(A_{24} - A_{13} - A_{23}) \\
\mathrm{CNONL(M,2)} &= (A_{24} - A_{13})r/R_2 \\
\mathrm{C(M,3)} &= r/R_2 + A_{14}r'^2/r + A_{13}r/R_1R_2 - A'_{13}r' \\
\mathrm{C(M,4)} &= A_{33}r \\
\mathrm{CNONL(M,3)} &= A'_{33}r + r'A_{33} \\
\mathrm{CNONL(M,4)} &= -A_{13}r/R_2 \\
\mathrm{CNONL(M,5)} &= r^2r'A_{23} \\
\mathrm{CNONL(M,6)} &= A'_{34}r' - A_{34}r/R_1R_2 - A_{44}r'^2/r \\
\mathrm{CNONL(M,7)} &= A'_{34}r/2R_2 + A_{34}r'/2R_1 - 3A_{44}r'/2R_2 \\
\mathrm{CNONL(M,8)} &= 2A_{14}r'/R_2 - A_{13}r'/R_1 - A'_{13}r/R_2 - r' \\
\mathrm{CNONL(M,9)} &= A_{14}/r - A_{13}(r'^2R_2/rR_1 + r/R_1R_2) - r/R_2 - r'^2R_2/r \\
\mathrm{CNONL(M,10)} &= (A_{24} - A_{13})(1 - 2r'^2)r + A'_{23}r'r^2 + A_{23}r(2r'^2 - r^2/R_1R_2) \\
\mathrm{F0(M)} &= r' + A_{13}r'/R_1 + A'_{13}r/R_2 - A_{14}r'/R_2 \\
\mathrm{F1(M)} &= r^2(A'_{23}\,r/R_2 + A_{23}r'(2/R_2 + 1/R_1) + A_{13}r'/R_2 - A_{24}r'/R_2)
\end{aligned} \tag{40}$$

The coefficients A_{ij} are obtained from a semi-inverted form of the constitutive equations:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \\ M_1 \\ M_2 \\ M_T \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & 0 & A_{23} & A_{24} & 0 \\ 0 & 0 & B_{33} & 0 & 0 & 0 \\ A_{31} & A_{32} & 0 & A_{33} & A_{34} & 0 \\ A_{41} & A_{42} & 0 & A_{43} & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} \quad (41)$$

The A_{ij} are easily calculated once the C_{ij} (Eqs. 1) are known.

The designations C(M,N), CNONL(M,N), FO(M) and Fl(M), (M = 2I+2 or 2I+1, I = 1, 2, 3, ...K + 1 and N = 1, 2, 3, ...) for the numerical coefficients are used in the computer program. The row number M denotes the m^{th} equation in the set of 2K + 6 finite difference equations generated by dividing the meridian into K intervals, satisfying equilibrium and compatibility at the K + 1 points in the domain and satisfying two boundary or symmetry conditions at either end of the meridian. Even equations (M = 2I + 2) are equilibrium equations and odd equations (M = 2I +1) are compatibility equations. The quantity $\Delta\psi$ is the difference between the total stress function ψ and the value of ψ calculated from membrane theory for uniform internal pressure:

$$\Delta\psi = \psi - rV*r'R_2/r \tag{42}$$

A star * denotes multiplication in the above and in the following equations. The computer program applies only to shells with constant properties along a meridian. Hence A'_{ij} are zero in the calculations.

In the prebuckling equilibrium problem the boundary conditions are expressed in the form

$$\begin{aligned}
BA11*\psi + BA12*M_1 + BA13*u_H + BA14*\beta &= \psi_A \\
BA21*\psi + BA22*M_1 + BA23*u_H + BA24*\beta &= M_{1A} \\
BB11*\psi + BB12*M_1 + BB13*u_H + BB14*\beta &= \psi_B \\
BB21*\psi + BB22*M_1 + BB23*u_H + BB24*\beta &= M_{1B}
\end{aligned} \tag{43}$$

Equations (43 a, b) are the boundary conditions at the end A of the meridian (see Fig. 5). Equations (43c, d) are the boundary conditions at B. The right hand sides are shown as stress functions and moments at A and B. However, they can be considered horizontal displacements and rotations. The boundary conditions are given in this general form in order to permit for instance treatment of composite shells in which the elastic properties of adjacent structures are accounted for through their stiffness coefficients. For cases in which the axial load is applied eccentrically with respect to the shell reference surface, the resulting bending moments $e_A \bar{V}_A$ and $e_B \bar{V}_B$ are included in M_{1A} and M_{1B}, respectively.

It is necessary to write Eqs. (43) in terms of ψ and β only. The boundary condition (43a), for example has the form

$$C(1,2)\psi' + C(1,3)\psi + C(1,5)\beta' + [CNONL(1,1) + CNONL(1,2)*\beta + CNONL(1,3)*\Delta\psi + CNONL(1,4)*p]\,\beta = FO(1)*rV + F1(1)*p + \psi_A \tag{44}$$

where

$$\begin{aligned}
C(1,2) &= -BA12*A_{23} + BA13*A_{22}r \\
C(1,3) &= BA11 - BA12*A_{13}r'/r + BA13*A_{12}r' \\
C(1,5) &= BA12*A_{33} + BA13*A_{23}r \\
CNONL(1,1) &= BA12*A_{34}r'/r + BA13*A_{24}r' + BA14 \\
CNONL(1,2) &= BA12*A_{34}/2R_2 + BA13*A_{24}r/2R_2 \\
CNONL(1,3) &= -BA12*A_{13}/R_2 + BA13*A_{12}r/R_2 \\
CNONL(1,4) &= BA12*A_{23}r'r - BA13*A_{22}r'r^2 \\
FO(1) &= BA12*A_{13}/R_2 - BA13*A_{12}r/R_2 \\
F1(1) &= BA12*A_{23}r^2/R_2 - BA13*A_{22}r^3/R_2
\end{aligned} \tag{45}$$

The row designation 1 indicates that Eq. (44) is the first in the set of 2K + 6 differential equations corresponding to equilibrium and compatibility at K + 1 points in the domain and 4 equations for the boundary conditions.

The method of finite differences is used to transform the governing differential equations into a set of nonlinear algebraic equations. Constant station spacing is used, and the derivatives of the dependent variables β and ψ are simulated by three-point finite difference formulas.

The nonlinear algebraic equations are solved by the Newton-Raphson method, of which an explanation is given in Ref. 6. The linear system which must be solved for each iteration in the Newton-Raphson process is characterized by a matrix of coefficients which is strongly banded about the main diagonal. This system is solved by efficient subroutines called FACTØR and SØLVE, which were written by Brogan (Ref. 7).

2.2 Special Prebuckling Analysis for Cylindrical Shells

In the case of general shells of revolution the prebuckling solution is obtained by use of a finite-difference analysis. The resulting nonlinear algebraic equations are solved through an iterative technique. For the cylindrical shell, however, the coefficients in the prebuckling equations are constant and thus an explicit analytical solution is readily available. For economy in the numerical analysis this solution is utilized.

For cylindrical shells, prebuckling equilibrium is governed by the equation

$$(M_1)_{xx} + \frac{1}{r} N_2 - N_1 w_{xx} - p = 0 \tag{46}$$

where

$$\begin{aligned} N_1 &= C_{11}\,\epsilon_1 + C_{12}\,\epsilon_2 + C_{14}\,\kappa_1 \\ N_2 &= C_{12}\,\epsilon_1 + C_{22}\,\epsilon_2 + C_{24}\,\kappa_1 \\ M_1 &= C_{14}\,\epsilon_1 + C_{15}\,\epsilon_2 + C_{44}\,\kappa_1 \end{aligned} \tag{47}$$

and

$$\epsilon_2 = w/r$$
$$\kappa_1 = w_{xx} \tag{48}$$

If an axial load N (positive for tension) is applied to the shell

$$N_1 = N \tag{49}$$

By combination of Eqs. (46-49) the equilibrium equation can be obtained in the form

$$w_{xxxx} - 4\,S\,w_{xx} + 4\,T^2 w = \xi \tag{50}$$

where

$$4\,S = \frac{[r\,C_{11}\,N + (2\,C_{12}C_{14} - C_{11}C_{24} - C_{11}C_{15})]}{r(C_{11}C_{44} - C_{14}{}^2)}$$

$$4\,T^2 = \frac{(C_{11}C_{22} - C_{12}^2)}{r^2\,(C_{11}C_{44} - C_{14}^2)}$$

$$\xi = \frac{p\,r\,C_{11} - N\,C_{12}}{r\,(C_{11}C_{44} - C_{14}^2)} \tag{51}$$

For the case $S + T > 0$ the solution of this equation is

$$w = \bar{w} + B_1 \cosh a_1x \cos a_2x + B_2 \sinh a_1x \sin a_2x + B_3 \cosh a_1x \sin a_2\,x$$
$$+ B_4 \sinh a_1x \cos a_2x \tag{52}$$

where

$$\bar{w} = r\,(r\,C_{11}p - C_{12}N)/(C_{11}C_{22} - C_{12}^2)$$

$$a_1 = (S - T)^{1/2}$$
$$a_2 = (S + T)^{1/2} \qquad (53)$$

In the case $S + T < 0$

$$w = \bar{w} + B_1 \cos a_1 x + B_2 \cos a_2 x + B_3 \sin a_1 x + B_4 \sin a_2 x \qquad (54)$$

where

$$a_1 = [2 \left\{ - S + (S^2 - T^2)^{1/2} \right\}]^{1/2}$$
$$a_2 = [2 \left\{ - S - (S^2 - T^2)^{1/2} \right\}]^{1/2} \qquad (55)$$

Depending on the sign of the discriminant $(S + T)$ one of the solutions Eqs. (52) or (54) is selected and substituted into the boundary conditions given by Eqs.(43). For cylindrical shells Eqs. (43) become

$$\begin{aligned}
BA11(rH)_A + BA12(M_1)_A + BA13(w)_A + BA14(w_x)_A &= \psi_A \\
BA21(rH)_A + BA22(M_1)_A + BA23(w)_A + BA24(w_x)_A &= (\bar{M}_1)_A \\
BB11(rH)_B + BB12(M_1)_B + BB13(w)_B + BB14(w_x)_B &= \psi_B \\
BB21(r\dot{H})_B + BB22(M_1)_B + BB23(w)_B + BB24(w_x)_B &= (\bar{M}_1)_B
\end{aligned} \qquad (56)$$

By use of Eqs. (47) and (48)

$$M_1 = (C_{44} - \frac{C_{14}^2}{C_{11}}) \; w_{xx} + \frac{C_{14}}{C_{11}} \; N - (C_{12}C_{14}/\, C_{11} - C_{15}) \frac{w}{r} \qquad (57)$$

Furthermore

$$rH = -r(M_1)_x + Nrw_x$$

$$= Nrw_x - [r\,(C_{44} - \frac{C_{14}^2}{C_{11}})\,w_{xxx} - (\frac{C_{12}C_{14}}{C_{11}} - C_{15})\,w_x] \qquad (58)$$

Eqs. (57) and (58) together with Eq. (51) or (54) are substituted into the boundary conditions (Eqs. 56). This results in a linear equation system from which the integration constants B_i are obtained

$$B_i = [X_{ij}]^{-1} \{Y_j\} \qquad (59)$$

For brevity the following notations are introduced

$$\bar{C}_{12} = (C_{12}\,C_{14} / C_{11} - C_{15})/r$$
$$\bar{C}_{44} = C_{44} - C_{14}^2/C_{11} \qquad (60)$$

		For $S + T > 0$	For $S + T < 0$
F_1	=	$\cosh(a_1L)\cos(a_2L)$	$\cos(a_1L)$
F_2	=	$\sinh(a_1L)\sin(a_2L)$	$\cos(a_2L)$
F_3	=	$\cosh(a_1L)\sin(a_2L)$	$\sin(a_1L)$
F_4	=	$\sinh(a_1L)\cos(a_2L)$	$\sin(a_2L)$

(61)

$$a_3 = a_1^2 - a_2^2 \qquad a_4 = 2\,a_1a_2$$
$$a_5 = a_2a_3 + a_1a_4 \qquad a_6 = a_1a_3 - a_2a_4 \qquad (62)$$

$G_1 = BB13 - BB12\ \bar{C}_{12}$ $H_1 = BB23 - BB_{22}\ \bar{C}_{12}$

$G_2 = BB14 + BB11\ r(N + \bar{C}_{12})$ $H_2 = BB24 + BB21\ r(N + \bar{C}_{12})$

$G_3 = BB12\ \bar{C}_{44}$ $H_3 = BB22\ \bar{C}_{44}$

$G_4 = r\ BB11\ \bar{C}_{44}$ $H_4 = r\ BB21\ \bar{C}_{44}$

For $S + T > 0$

$K_1 = a_3\ \bar{C}_{44} + \bar{C}_{12}$

$K_2 = r\ [a_2\ (N + \bar{C}_{12}) - a_5\ \bar{C}_{44}]$

$K_3 = r\ [a_1\ (N + \bar{C}_{12}) - a_6\ \bar{C}_{44}]$

For $S + T < 0$

$K_1 = a_1^2\ \bar{C}_{44} + \bar{C}_{12}$

$K_2 = a_2^2\ \bar{C}_{44} + \bar{C}_{12}$

$K_3 = r\ (N + \bar{C}_{12} - a_1^2\ \bar{C}_{44})$

$K_4 = r\ (N + \bar{C}_{12} - a_2^2\ \bar{C}_{44}$

The integration constants B_i in equations (51) and (54) can be determined from Eq. (59). The coefficients X_{ij} and Y_j in this equation are

For S + T > 0

$$X_{11} = BA13 + BA12\ K_1 \qquad X_{21} = BA23 + BA22\ K_1$$

$$X_{12} = BA12\ a_4\ \bar{C}_{44} \qquad X_{22} = BA22\ a_4\ \bar{C}_{44}$$

$$X_{13} = BA14\ a_2 + BA11\ K_2 \qquad X_{23} = BA24\ a_2 + BA21\ K_2$$

$$X_{14} = BA14\ a_1 + BA11\ K_3 \qquad X_{24} = BA24\ a_1 + BA21\ K_3$$

$$X_{31} = F_1G_1 + (a_1F_4 - a_2F_3)\ G_2 + (a_3F_1 + a_4F_2)G_3 + (a_6F_4 - a_5F_3)G_4$$

$$X_{32} = F_2G_1 + (a_2F_4 + a_1F_3)\ G_2 + (a_4F_1 + a_3F_2)G_3 + (a_5F_4 + a_6F_3)G_4$$

$$X_{33} = F_3G_1 + (a_1F_2 + a_2F_1)\ G_2 + (a_3F_3 + a_4F_4)G_3 + (a_5F_1 + a_6F_2)G_4$$

$$X_{34} = F_4G_1 + (a_1F_1 - a_2F_2)\ G_2 + (a_3F_4 - a_4F_3)G_3 + (a_6F_1 - a_5F_2)G_4$$

$$X_{41} = F_1H_1 + (a_1F_4 - a_2F_3)\ H_2 + (a_3F_1 + a_4F_2)H_3 + (a_6F_4 - a_5F_3)H_4$$

$$X_{42} = F_2H_1 + (a_2F_4 + a_1F_3)\ H_2 + (a_4F_1 + a_3F_2)H_3 + (a_5F_4 + a_6F_3)H_4$$

$$X_{43} = F_3H_1 + (a_1F_2 + a_2F_1)\ H_2 + (a_3F_3 + a_4F_4)H_3 + (a_5F_1 + a_6F_2)H_4$$

$$X_{44} = F_4H_1 + (a_1F_1 - a_2F_2)\ H_2 + (a_3F_4 - a_4F_3)H_3 + (a_6F_1 - a_5F_2)H_4$$

$$\begin{aligned}
Y_1 &= -\ BA13\ \bar{w} - BA12\ (\bar{w}\ \bar{C}_{12} + N\ C_{14}) + \psi_A \\
Y_2 &= -\ BA23\ \bar{w} - BA22\ (\bar{w}\ \bar{C}_{12} + N\ C_{14}) + (\bar{M}_1)_A - e_A\ N \\
Y_3 &= -\ \bar{w}\ G_1 - BB12\ N\ C_{14} + \psi_B \\
Y_4 &= -\ \bar{w}\ H_1 - BB22\ N\ C_{14} + (\bar{M}_1)_B - e_B\ N
\end{aligned} \qquad (65)$$

For $S + T < 0$

$$
\begin{aligned}
X_{11} &= BA13 - BA12\ K_1 & X_{21} &= BA23 - BA22\ K_1 \\
X_{12} &= BA13 - BA12\ K_2 & X_{22} &= BA23 - BA22\ K_2 \\
X_{13} &= a_1\ (BA14 + BA11\ K_3) & X_{23} &= a_1\ (BA24 + BA21\ K_3) \\
X_{14} &= a_2\ (BA14 + BA11\ K_4) & X_{24} &= a_2\ (BA24 + BA21\ K_4)
\end{aligned}
$$

$$
\begin{aligned}
X_{31} &= G_1 F_1 - G_2 a_1 F_3 - G_3 a_1^2 F_1 + G_4 a_1^3 F_3 \\
X_{32} &= G_1 F_2 - G_2 a_2 F_4 - G_3 a_2^2 F_2 + G_4 a_2^3 F_4 \\
X_{33} &= G_1 F_3 + G_2 a_1 F_1 - G_3 a_1^2 F_3 - G_4 a_1^3 F_1 \\
X_{34} &= G_1 F_4 + G_2 a_2 F_2 - G_3 a_2^2 F_4 - G_4 a_2^3 F_2 \\
X_{41} &= H_1 F_1 - H_2 a_1 F_3 - H_3 a_1^2 F_1 + H_4 a_1^3 F_3 \\
X_{42} &= H_1 F_2 - H_2 a_2 F_4 - H_3 a_2^2 F_2 + H_4 a_2^3 F_4 \\
X_{43} &= H_1 F_3 + H_2 a_1 F_1 - H_3 a_1^2 F_3 - H_4 a_1^3 F_1 \\
X_{44} &= H_1 F_4 + H_2 a_2 F_2 - H_3 a_2^2 F_4 - H_4 a_2^3 F_2
\end{aligned}
$$

$$
\begin{aligned}
Y_1 &= -\bar{w}\ (BA13 - BA12\ \bar{C}_{12}) - BA12\ N\ C_{14}/C_{11} + \psi_A \\
Y_2 &= -\bar{w}\ (BA23 - BA22\ \bar{C}_{12}) - BA22\ N\ C_{14}/C_{11} + (\bar{M}_1)_A - e_A\ N \\
Y_3 &= -\bar{w}\ G_1 - BB12\ N\ C_{14}/C_{11} + \psi_B \\
Y_4 &= -\bar{w}\ H_1 - BB22\ N\ C_{14}/C_{11} + (\bar{M}_1)_B - e_B\ N
\end{aligned}
\tag{66}
$$

Section 3

STABILITY ANALYSIS

3.1 Stability Equations

Donnell-type equations are used in the stability analysis. These equations are based on the following assumptions:

1. Love's first approximation
2. the flexural and extensional strains are of comparable magnitude
3. the shortest wavelength of deformations small in comparison to the minimum radius of curvature

The first assumption is basic to almost every engineering analysis of shells. The second assumption has two important consequences: (1) Two of the three compatibility equations for the deformed middle surface may be approximated by the similar equations for an inextensional deformation of the middle surface, and (2) a moment resultant divided by a radius of curvature of the middle surface, or multiplied by a change in curvature can be neglected compared to the normal stress resultants. The third assumption permits the approximate solution of the in-plane equilibrium equation by an Airy-type stress function.

The Donnell-type equations are used here rather than more exact stability equations for the following reasons:

1. A stress function φ can be introduced, which leads to a reduction in the number of dependent variables in the analysis. In the Donnell-type analysis the two dependent variables are φ and w ; in a more exact analysis the three dependent variables are u, v, and w. The consequent reduction in the number of finite-difference equations leads to reduction in the computer core storage and time required for calculation of the buckling load.

2. The Donnell-type analysis does not include derivatives of the radius of curvature of the shell. Such data as R_1' , R_1'' , and R_1''' are needed as input for the more exact analysis. These quantities are difficult to obtain for shells whose meridians are not defined by analytical expressions.

3. The Donnell-type analysis yields sufficiently accurate results for almost all engineering applications. It is not accurate when applied to shells which buckle in an almost inextensional mode, such as shells with weak support at the edges. However, in most engineering applications, inextensional buckling modes may be eliminated by proper design of the structure which supports the shell.

More exact stability equations, which are not restricted by all of the assumptions basic to the Donnell-type of analysis, are given in Appendix A.

In a Donnell type analysis the governing equilibrium equations are:

$$
\begin{aligned}
&(rN_1)' - r'N_2 + N_{12}^{\cdot} = 0 \\
&N_2^{\cdot} + (rN_{12})' + r'N_{12} = 0 \\
&-(rM_1)'' - M_2^{\cdot\cdot}/r + (r'M_2)' + M_T^{\prime\cdot} + r'M_T^{\cdot}/r \\
&\qquad = r(N_1k_1^* + N_2k_2^* - N_{10}\varkappa_1 - N_{20}\varkappa_2)
\end{aligned}
\tag{67}
$$

and the compatibility equations are

$$
\begin{aligned}
&(r\varkappa_2)' - r'\varkappa_1 + \varkappa_{12}^{\cdot} = 0 \\
&\varkappa_1^{\cdot} + (r\varkappa_{12})' + r'\varkappa_{12} = 0 \\
&-(r\epsilon_2)'' - \epsilon_1^{\cdot\cdot}/r + (r'\epsilon_1)' + \epsilon_{12}^{\prime\cdot} + r'\epsilon_{12}^{\cdot}/r \\
&\qquad = -r(\varkappa_2k_1^* + \varkappa_1k_2^* + \beta_o Kw')
\end{aligned}
\tag{68}
$$

Equations (67) and (68) are almost analogous; one obtained from the other through

$$
\begin{aligned}
&\varkappa_2 \to N_1 \qquad \varkappa_1 \to N_2 \qquad \varkappa_{12} \to N_{12} \\
&\epsilon_2 \to M_1 \qquad \epsilon_1 \to M_2 \qquad \epsilon_{12} \to M_T
\end{aligned}
\tag{69}
$$

Equations (67a) and (67b) and Eqs. (68a) and (68b) are satisfied approximately by the following Airy-type stress and curvature functions:

$$
\begin{aligned}
N_1 &= \ddot{\varphi}/r^2 + \varphi' r'/r \\
N_2 &= \varphi'' \\
N_{12} &= -(\dot{\varphi}/r)' \\
\varkappa_1 &= w'' \\
\varkappa_2 &= \ddot{w}/r^2 + w' r'/r \\
\varkappa_{12} &= -(\dot{w}/r)'
\end{aligned}
\tag{70}
$$

According to Koiter (Ref. 8) w can be considered a "curvature function", analogous to a stress function. There is no need to define it as the normal displacement. However, in the present investigation, where the displacements from the prebuckled equilibrium state are considered to be infinitesimal, w does represent the actual normal displacement within the accuracy of the "shallow" shell equations.

The final governing equations are written in terms of φ and w by insertion of Eqs. (70) into (67c) and (68c), with the use of the constitutive equations (41).

Two ordinary differential equations of fourth order result when the relations

$$\begin{aligned} \varphi &= \varphi_n(s) \sin n\theta \\ w &= w_n(s) \sin n\theta \end{aligned} \tag{71}$$

are inserted into the partial differential equations of compatibility and equilibrium. The final equations (two equations for each meridional station as explained above) have the form:

$$\begin{aligned} &CB(M,1)\varphi^{iv} + CB(M,2)\varphi''' + CB(M,3)\varphi'' + CB(M,4)\varphi' + CB(M,5)\varphi \\ &\quad + CB(M,6)w^{iv} + CB(M,7)w''' + CB(M,8)w'' + CB(M,9)w' + CB(M,10)w = 0 \end{aligned} \tag{72}$$

The coefficients of φ and its derivatives in the compatibility equation are

$$\begin{aligned} CB(M,1) &= -r A_{22} \\ CB(M,2) &= r'(A_{12} - A_{21} - 2A_{22}) - 2rA_{22}' \\ CB(M,3) &= r'^2 A_{11}/r + n^2(A_{12} + A_{21} + B_{33})/r + rK(A_{22} - A_{12} + 2A_{21}) \\ &\qquad - 2r'A_{21}' - rA_{22}'' - 2r'A_{22}' + r'A_{12}' + f_1 \end{aligned}$$

$$\begin{aligned} CB(M,4) &= -A_{11}(r'^3/r^2 + 2r'K) + A_{21}(r'K - 2n^2 r'/r^2) \\ &\qquad - B_{33} n^2 r'/r^2 + A_{21}'(2rK + n^2/r - r'^2/r) - r'A_{21}'' \\ &\qquad + A_{11}' r'^2/r + n^2 B_{33}'/r + f_2 \end{aligned}$$

$$
\begin{aligned}
CB(M,5) \;=\; & n^2\Big\{A_{11}(2r'^2/r^2 + K - n^2/r^2) + A_{21}(K + 2r'^2/r^2) \\
& + B_{33}(K + r'^2/r^2)\Big\}/r + n^2A''_{21}/r - r'A'_{11}n^2/r^2 \\
& - r'B'_{33}\, n^2/r^2 + f_3
\end{aligned} \tag{73}
$$

The coefficients of the w-terms in the compatibility equation and of all the terms in the equilibrium equation can be obtained through modification of Eqs. (73) as shown in Table 1. Table 1 gives the changes which thereby must be made in the indicies of the A_{ij} and in the definitions of f_1, f_2, and f_3.

The designation CB(M,N) for the coefficients is also used in the computer program. The row number M denotes the m^{th} equation in the set of 2K + 10 finite-difference equations generated by division of the meridian into K intervals, satisfaction of compatibility and equilibrium at the K + 1 points in the domain, and satisfaction of 4 boundary or symmetry conditions at each end of the meridian. In the computer program odd equations are compatibility equations and even equations are equilibrium equations. The computer program is specialized for shells whose properties are constant along a meridian. Hence, in all calculations $A'_{ij} = A''_{ij} = B'_{kk} = 0$.

- Boundary Conditions

The formulation of the stability analysis must be compatible with the formulation of the prebuckling analysis. Therefore, the boundary conditions

Table 1

Changes in A_{ij} and Definitions of f_k for the Coefficients CB(M,N) of the Compatibility and Equilibrium Equations

Compatibility		Equilibrium	
φ-terms CB(M,N), N = 1 to 5 replace	w-terms CB(M,N), N = 6 to 10 by	φ-terms CB(M,N), N = 1 to 5 by	w-terms CB(M,N), N = 6 to 10 by
N	N + 5	N	N + 5
A_{11}	A_{14}	A_{41}	A_{44}
A_{12}	A_{13}	A_{42}	A_{43}
A_{21}	A_{24}	A_{31}	A_{34}
A_{22}	A_{23}	A_{32}	A_{33}
B_{33}	0	0	B_{66}
$f_1 = 0$	$f_1 = rk_2^*$	$f_1 = -rk_2^*$	$f_1 = rN_{10}$
$f_2 = 0$	$f_2 = r'k_1^* + \beta_o rK$	$f_2 = -\ r'k_1^*$	$f_2 = r'N_{20}$
$f_3 = 0$	$f_3 = -n^2k_1^*/r$	$f_3 = n^2k_1^*/r$	$f_3 = -n^2N_{20}/r$

for the stability analysis are treated in a manner analogous to that explained in the section on prebuckling analysis. Boundary conditions are written in the form

$$\begin{bmatrix} BA11 & BA12 & BA13 & BA14 & BA15 & BA16 & BA17 & BA18 \\ BA21 & BA22 & BA23 & BA24 & BA25 & BA26 & BA27 & BA28 \\ BA31 & BA32 & BA33 & BA34 & BA35 & BA36 & BA37 & BA38 \\ BA41 & BA42 & BA43 & BA44 & BA45 & BA46 & BA47 & BA48 \\ BB11 & BB12 & BB13 & BB14 & BB15 & BB16 & BB17 & BB18 \\ BB21 & BB22 & BB23 & BB24 & BB25 & BB26 & BB27 & BB28 \\ BB31 & BB32 & BB33 & BB34 & BB35 & BB36 & BB37 & BB38 \\ BB41 & BB42 & BB43 & BB44 & BB45 & BB46 & BB47 & BB48 \end{bmatrix} \begin{Bmatrix} rH \\ M_1 \\ u_H \\ \beta \\ v \\ rN_{12} \\ u_v \\ rV \end{Bmatrix} = 0 \tag{74}$$

The first 4 of Eqs. (74) are the boundary conditions at A (see Fig. 5). The last 4 are the boundary conditions at B. The boundary conditions are expressed in the form given by Eqs. (74) in order to permit the treatment of shells bounded by elastic rings as well as shells for which more simple boundary conditions are assumed. From Eqs. (74) it can be seen for example that "simple-support" conditions at the ends A and B of the meridian can be simulated by the specification BA13 = 1.0, BA22 = 1.0, BA35 = 1.0, BA48 = 1.0, for the non-zero coefficients at A; and BB13 = 1.0, BB22 = 1.0, BB35 = 1.0, and BB48 = 1.0 for the non-zero coefficients at B. All other BAij's and BBij's are set equal to zero for this case. Physically the above-specified input parameters correspond respectively to $u_H = 0$, $M_1 = 0$, $v = 0$, and $rV = 0$ at A and at B.

If the shell is supported by elastic rings at the boundaries, the BAij and BBij are computed from ring equations given by Cheney (Ref. 9). Formulas for these coefficients are presented in Ref. 10 and will not be repeated here. They are valid for any ring, the centroid of which coincide with its shear center. The ring centroid need not coincide with the shell reference surface. It should be emphasized that the BAij and BBij corresponding to the boundary conditions (Eqs. 43) of the prebuckling equilibrium problem (wave number $n = 0$) are not the same as those corresponding to the boundary conditions (Eqs. 74) of the stability problem ($n \neq 0$). The effective stiffness of the ring depends on the wave number n .

When the quantities appearing in the column vector of Eqs. (74) are expressed in terms of φ and w, the boundary condition equations are of the same form as the equilibrium and compatibility equations, (Eq. 72). The expressions for u and v in terms of φ and w are obtained by elimination of v and the s-derivatives of u from the strain-displacement relations

$$\begin{aligned} \epsilon_1 &= u' + w/R_1 + \beta_o w' \\ \epsilon_2 &= ur'/r + v^{\cdot}/r + w/R_2 \\ \epsilon_{12} &= u^{\cdot}/r + r(v/r)' + \beta_o w^{\cdot}/r \end{aligned} \tag{75}$$

The following equation is found:

$$u^{\cdot\cdot}/r + u(rK + r'^2/r) = \dot{\epsilon}_{12} - r\epsilon_2' + r'\epsilon_1 + rw'k_2^* - r'w/R_2 - \beta_o w^{\cdot\cdot}/r \tag{76}$$

The circumferential displacement v can be found from Eq. (75b) and Eq. (41).

The forces and displacements normal and tangential to the middle surface can be written in terms of φ and w. Table 2 gives the coefficients of φ and its derivatives. The quantities u_1, u_2, u_3, and u_4 in the sixth row (v) represent the corresponding coefficients shown in the fifth row (u). With regard to the A_{ij} and B_{kk}, the changes shown in Table 1 must be made in order to derive by analogy the coefficients of the w-terms. The functions f_4 through f_{10} are given in Table 3.

3.2 Solution of the Equations

The method of finite differences is used to solve the linear differential equations of the form (72). The set of finite difference equations is arranged as described in the section on prebuckling analysis. Hence the matrix of coefficients is strongly banded about the main diagonal.

The derivatives of φ and w are simulated by 5-point central difference formulas and the coefficients of $2M + 8$ algebraic equations are stored in a condensed matrix A. The $2M + 8$ equations correspond to the compatibility equation and the equilibrium equation at M points on the meridian, and 8 boundary conditions, four at each end of the shell. The stability equations for $n = 0$ are derived by appropriate modification of the nonlinear prebuckling equations. These stability equations are given in Ref.11 and will not be repeated here.

Table 2 Coefficients for Tangential and Normal Forces and Displacements when $n \neq 0$

	φ'''	φ''	φ'	φ
rN_1			r'	$-n^2/r$
rQ	$-rA_{32}$	$r'(A_{42}-A_{31}-A_{32})$ $-rA_{32}'$	$r'^2A_{41}/r - r'A_{31}' + A_{31}(rK+n^2/r) + f_4$	$-r'n^2(A_{41} + A_{31})/r^2 + n^2A_{31}'/r + f_5$
rN_{12}			$-n$	nr'/r
M_1		A_{32}	$A_{31}r'/r$	$-A_{31}n^2/r^2$
u	$-rA_{22}/F$	$-rA_{22}'/F$ [a]	$\left\{A_{11}r'^2/r + B_{33}n^2/r + A_{21}(r^2K+n^2+r'^2)/r - r'A_{21}' + f_6\right\}/F$	$\left\{-r'n^2(A_{11} + 2A_{21} + B_{33})/r^2 + n^2A_{21}'/r + f_7\right\}/F$
v	$r'u_1/n$	$(r'u_2-rA_{22})/n$	$r'u_3 - r'A_{21})/n$	$(r'u_4 + A_{21}n^2/r)/n + f_8$
w				f_9
β			f_{10}	

[a] $F = rK + r'^2/r - n^2/r$

Table 3

Definitions of f_k for the Boundary Conditions

φ - terms	w - terms
$f_4 = \beta_o r'$	$f_4 = rN_{10} + B_{66}n^2/r$
$f_5 = -n^2\beta_o/r$	$f_5 = -n^2 r' B_{66}/r^2$
$f_6 = 0$	$f_6 = rk_2^*$
$f_7 = 0$	$f_7 = -r'/R_2 + n^2\beta_o/r$
$f_8 = 0$	$f_8 = 1.0$
$f_9 = 0$	$f_9 = r/(nR_2)$
$f_{10} = 0$	$f_{10} = 1.0$

The stability equations in finite-difference form are a set of linear, homogeneous, algebraic equations. There exist nontrivial solutions of this set for discrete values of a parameter, in this case a load parameter. The lowest eigenvalue is the buckling load. Its value can be obtained by various methods. The determinant of the coefficient matrix can be plotted versus the load in order to find the point where its sign first changes. On the other hand, when the eigenvalue problem has the form

$$(A + \lambda B)x = 0 \tag{77}$$

an iteration scheme (Ref. 12) can be employed to calculate both the lowest eigenvalue and the corresponding eigenvector.

In this analysis in which nonlinear prebuckling effects are included, the eigenvalue problem does not have the simple form of Eq. (77). The eigenvalue parameter λ does not appear linearly,but manifests itself through its influence on the prebuckling meridional rotation β_o, stress resultants N_{10} and N_{20}, and changes in curvature κ_{10} and κ_{20} , which appear in the coefficients of the stability equations. These quantitites are related in a nonlinear way to the loading. However, there are many practical shell structures which buckle when N_{10} and N_{20} are very close to the values predicted from membrane theory and when β_o is so small that linear membrane theory is still accurate. In such cases the stability equations can be approximated by equations of the form of Eq. (77) and the power method (Ref. 12) can be used to find the lowest eigenvalue and corresponding eigenvector with a fair degree of accuracy.

When the prebuckling bending stress and other nonlinear prebuckling terms are important, such as in the case of bifurcation buckling of externally pressurized shallow spherical caps, the power method can be used to advantage in the following way: The determinant D of coefficients of the stability equations is evaluated for increasing values of the loading parameter ρ . Figure 6 shows a plot of D versus ρ . There is a ρ-interval, ρ_1 to ρ_2 , in which D first changes sign. The load for which D = 0 is approximated by linear interpolation from the end-points ρ_1 and ρ_2 of the interval. The error in the buckling load is now $\Delta\rho = \rho_3 - \rho_{cr}$. If $\Delta\rho/\rho_{cr} \ll 1$, then β_o , N_{10} and N_{20} can be expanded in Taylor series about ρ_3:

$$\begin{aligned} \beta_o &= (\beta_o)_3 + z(d\beta_o/d\rho)_3 \\ N_{10} &= (N_{10})_3 + z(dN_{10}/d\rho)_3 \\ N_{20} &= (N_{20})_3 + z(dN_{20}/d\rho)_3 \end{aligned} \tag{78}$$

where $z = \rho - \rho_3$. There are similar expansions for the prebuckling changes in curvature $\varkappa_{10}$ and $\varkappa_{20}$. The derivatives $d\beta_o/d\rho$, etc. are calculated from interpolation formulas such as:

$$\left(\frac{dN_{10}}{d\rho}\right)_3 = \frac{L_1(N_{10})_2}{L_2\ \mathrm{STEP}} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)(N_{10})_3 - \frac{L_2(N_{10})_1}{L_1\ \mathrm{STEP}} \tag{79}$$

Subscripts 1, 2, and 3 refer to values corresponding to ρ_1 , ρ_2 , and ρ_3 . The quantities L_1 , L_2 , and STEP are shown in Fig. 6. A value for z is calculated from the linear system

$$(A + zB)x = 0 \tag{80}$$

through use of the power method. Since $\Delta\rho/\rho_{cr} \ll 1$, convergence is indeed rapid. The new value of ρ is $\rho = \rho_3 + z$. If $|z/\rho_3|$ is less than some preassigned number ERR, calculations for the eigenvalue terminate. If not, new derivatives $(d\beta_o/d\rho)_4$, etc. are calculated, and a new correction factor is calculated from Eq. (80). Iterations proceed until $|z/\rho| < \text{ERR}$.

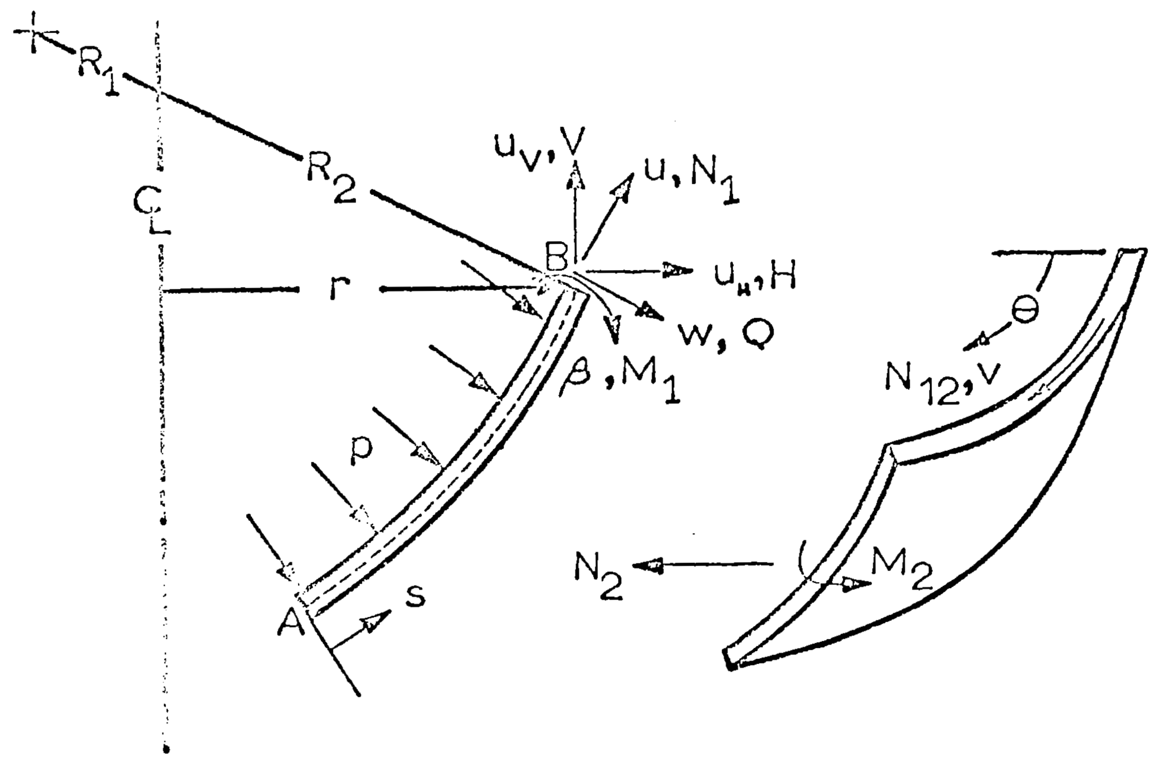

FIG. 5 - COORDINATES, GEOMETRY, LOADS, DISPLACEMENTS & STRESSES

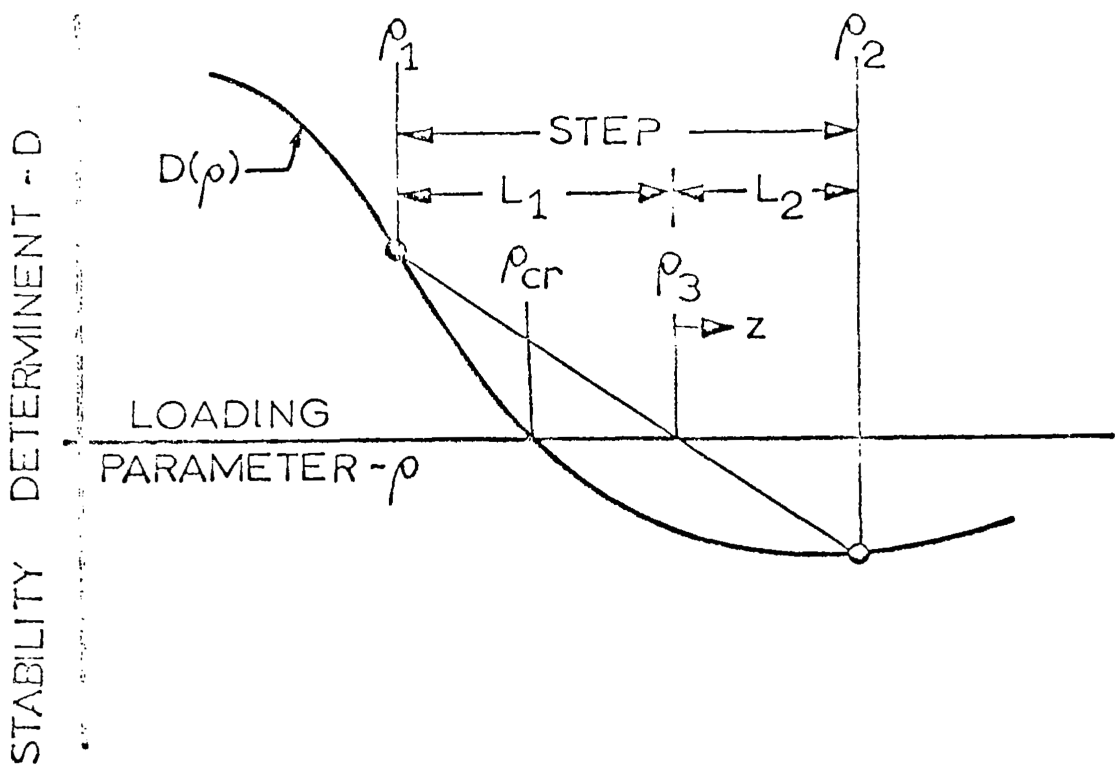

FIG. 6 - DETERMINANT PLOT

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Appendix A
MORE GENERAL STABILITY EQUATIONS

NOTATION

D_{ij}	coefficients of constitutive equations (see Eqs. A2)
H	horizontal (radial) force/length (see Fig. 5)
n	number of circumferential waves in buckle pattern
$N_x, N_\theta, N_{x\theta} = N_{\theta x}$ $M_x, M_\theta, M_{x\theta} = M_{\theta x}$ Q_x, Q_θ	incremental stress resultants (see Fig. 9)
$N_{x0}, N_{\theta 0}$	prebuckling stress resultants
p	normal pressure, positive outward
r	radius of a parallel circle
r_1, r_2	meridional and circumferential radii of curvature (see Fig. 7)
u, v, w	circumferential, meridional, and normal displacement components (see Fig. 8)
u_H	horizontal (radial) displacement (see Fig. 5)
u_V	vertical (axial) displacement (see Fig. 5)
V	vertical (axial) force/length (see Fig. 5)
x	arc length, meridional coordinate (see Fig. 7)
z	normal coordinate (see Fig. 9)
γ	$= \dfrac{\dot{r}}{r}$

$\bar{\epsilon}_x$, $\bar{\epsilon}_\theta$, $\bar{\gamma}_{x\theta}$	middle surface strains
κ_x, κ_θ, $\kappa_{x\theta}$	curvature changes
ρ, ρ_1, ρ_2	$= \frac{1}{r}, \frac{1}{r_1}, \frac{1}{r_2}$
θ, φ	circumferential and meridional coordinates (see Fig. 7)
ω_θ, ω_x ω_{z1}, ω_{z2}	rotation components (see Fig. 8 and Eqs. A3)
$\omega_{\theta 0}$	prebuckling rotation

Subscripts and superscripts

$\dot{(\ )}$	differentiation with respect to arc length x
$(\ )_x$	pertains to meridional direction
$(\ )_\theta$	pertains to circumferential direction
$(\ )_0$	prebuckling quantity

A1 Introduction

As was discussed in Section 3, Donnell-type stability equations give reasonably good solutions for almost all shell buckling problems encountered in practice. However, more accurate stability equations are needed if the shell buckles in an almost inextensional mode. Sobel (Ref. 13) has derived more general stability equations for axisymmetrically loaded isotropic shells of revolution of constant thickness. These equations are extended here to cover orthotropic shells of variable thickness. The resulting equations are identical to those derived by Kempner (Ref. 14) except for terms involving rotations around the normal which were not considered in Kempner's analysis. Three stability equations in terms of the three displacement components u, v, and w are obtained through combination of the equilibrium equations with the constitutive and kinematic relations. These equations are presented in Section A2.

It may be desirable to investigate the effect of certain terms, such as rotations around the normal. Therefore the following parameters are introduced in the final equations:

$$\delta_{RN} = \begin{cases} 1, & \text{if effects of terms involving rotation around the normal are included} \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{\omega} = \begin{cases} 1, & \text{if the prebuckling rotation is included in the equilibrium equations} \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{EL} = \begin{cases} 1, & \text{if the prebuckling rotation is included in the} \\ & \text{kinematic equations} \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{BC} = \begin{cases} 1, & \text{if the prebuckling rotation is included in the} \\ & \text{boundary condition for } Q_x \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{ph} = \begin{cases} 1, & \text{if the normal pressure } p \text{ is hydrostatic} \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{d} = \begin{cases} 1, & \text{if the stability equations are to be specialized to} \\ & \text{Donnell-type equations} \\ 0, & \text{otherwise} \end{cases}$$

A2 Stability Equations (Ref. 13)

- Equilibrium Equations

$$
\begin{aligned}
\sum F_x = {} & \dot{N}_x + \gamma(N_x - N_\theta) + n\rho N_{\theta x} + \delta_d\{ - \rho_1\dot{M}_x - \gamma\rho_1(M_x - M_\theta) \\
& - n\rho\rho_1 M_{x\theta} - \rho_1 N_{x0}\omega_\theta - \delta_\omega \rho_1 N_x \omega_{\theta 0} + \delta_{ph} p\omega_\theta - \delta_{RN} n\rho N_{\theta 0}\omega_{z1}\} = 0 \\[2ex]
\sum F_\theta = {} & 2\gamma N_{x\theta} + \dot{N}_{x\theta} - n\rho N_\theta + \delta_d\{-2\gamma\rho_2 M_{x\theta} - \rho_2\dot{M}_{x\theta} \\
& + n\rho\rho_2 M_\theta - \rho_2 N_{\theta 0}\omega_x - \delta_\omega \rho_2 N_{x\theta}\omega_{\theta 0} + \delta_{ph} p\omega_x \\
& - \delta_{RN}\gamma N_{\theta 0}\omega_{z1} + \delta_{RN}(\gamma N_{x0} + \dot{N}_{x0})\omega_{z2} + \delta_{RN} N_{x0}\dot{\omega}_{z2}\} = 0 \\[2ex]
\sum F_z = {} & \rho_1 N_x + \rho_2 N_\theta - n^2\rho^2 M_\theta + 2n\gamma\rho M_{x\theta} + 2n\rho M_{x\theta} \\
& - \rho_1\rho_2(M_x - M_\theta) + \gamma(2\dot{M}_x - \dot{M}_\theta) + \ddot{M}_x + \delta_\omega(\gamma N_x + \dot{N}_x)\omega_{\theta 0} \\
& + \delta_\omega N_x\dot{\omega}_{\theta 0} + \delta_\omega n\rho N_{x\theta}\omega_{\theta 0} + (\gamma N_{x0} + \dot{N}_{x0})\omega_\theta + N_{x0}\dot{\omega}_\theta \\
& + n\rho N_{\theta 0}\omega_x - \delta_{ph} p(\bar{\epsilon}_x + \bar{\epsilon}_\theta) = 0
\end{aligned}
\tag{A1}
$$

- Constitutive Equations

$$
\begin{bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{bmatrix}
=
\begin{bmatrix}
D_{11} & D_{12} & 0 & D_{14} & D_{15} & 0 \\
D_{12} & D_{22} & 0 & D_{15} & D_{25} & 0 \\
0 & 0 & D_{33} & 0 & 0 & D_{36} \\
D_{14} & D_{15} & 0 & D_{44} & D_{45} & 0 \\
D_{15} & D_{25} & 0 & D_{45} & D_{55} & 0 \\
0 & 0 & D_{36} & 0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_\theta \\ \bar{\gamma}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\kappa_{x\theta} \end{bmatrix}
\tag{A2}
$$

- Kinematic Equations

$$
\begin{aligned}
\bar{\epsilon}_x &= \dot{v} + \rho_1 w + \omega_{00}\omega_\theta \delta_{EL} & \omega_\theta &= -\dot{w} + \rho_1 v \\
\bar{\epsilon}_\theta &= n\rho u + \gamma v + \rho_2 w & \omega_x &= n\rho w + \rho_2 u \\
\bar{\gamma}_{x\theta} &= -\gamma u + \dot{u} - n\rho v + \omega_{00}\omega_x \delta_{EL} & \omega_{z1} &= n\rho v \\
\kappa_x &= (-\dot{\rho}_1 v - \rho_1 v)\delta_d + \ddot{w} & \omega_{z2} &= \dot{u} - \gamma u \\
\kappa_\theta &= (-n\rho\rho_2 u - \gamma\rho_1 v)\delta_d - n^2\rho^2 w + \gamma w \\
2\kappa_{x\theta} &= (-\gamma\rho_1 u + 2\gamma\rho_2 u - \rho_2 \dot{u} + n\rho\rho_1 v)\delta_d + 2n\gamma\rho w - 2n\rho w
\end{aligned}
\tag{A3}
$$

- Stability Equations Written in Terms of Displacements

Insertion of Eqs. (A 2) and (A 3) into Eqs. (A 1) results in the following stability equations for a variable thickness, orthotropic shell of revolution subjected to axially symmetric loads:

$$a_1\ddot{v} + a_2\dot{v} + a_3 v + a_4\dot{u} + a_5 u + a_6\dddot{w} + a_7\ddot{w} + a_8\dot{w} + a_9 w = 0$$

$$b_1\dot{v} + b_2 v + b_3\ddot{u} + b_4\dot{u} + b_5 u + b_6\ddot{w} + b_7\dot{w} + b_8 w = 0$$

$$c_1\dddot{v} + c_2\ddot{v} + c_3\dot{v} + c_4 v + c_5\ddot{u} + c_6\dot{u} + c_7 u$$

$$+ c_8\ddddot{w} + c_9\dddot{w} + c_{10}\ddot{w} + c_{11}\dot{w} + c_{12} w = 0$$

where

$$a_i = \tilde{a}_i - \bar{\bar{a}}_i - \bar{a}_i - \hat{a}_i - \overset{*}{a}_i \quad , \quad i = 1, \ldots, 9$$

$$b_i = \tilde{b}_i - \bar{\bar{b}}_i - \bar{b}_i - \hat{b}_i - \overset{*}{b}_i \quad , \quad i = 1, \ldots, 8$$

$$c_i = \tilde{c}_i - \bar{\bar{c}}_i - \bar{c}_i - \hat{c}_i - \overset{*}{c}_i \quad , \quad i = 1, \ldots, 12$$

and the non-zero values of these coefficients are

$$\tilde{a}_1 = d_1 - d_3\rho_1\delta_d$$

$$\tilde{a}_2 = d_2\gamma - 2d_3\dot{\rho}_1\delta_d - d_4\gamma\rho_1\delta_d + d_5 - d_7\rho_1\delta_d$$

$$\tilde{a}_3 = d_2\dot{\gamma} - d_3\ddot{\rho}_1\delta_d - d_4(\gamma\rho_1)^{\cdot}\delta_d + d_6\gamma - d_7\dot{\rho}_1\delta_d - d_8\gamma\rho_1\delta_d - n\rho(d_9 - d_{10}\rho_1\delta_d)$$

$$\tilde{a}_4 = n\rho(d_2 - d_4\rho_2\delta_d) + d_9 - d_{10}\rho_2\delta_d$$

$$\tilde{a}_5 = n[d_2\dot{\rho} - d_4(\rho\rho_2)^{\cdot}\delta_d + d_6\rho - d_8\rho\rho_2\delta_d] - \gamma[d_9 - d_{10}(2\rho_2 - \rho_1)\delta_d]$$

$$\tilde{a}_6 = d_3$$

$$\tilde{a}_7 = d_4\gamma + d_7$$

$$\tilde{a}_8 = d_1\rho_1 + d_2\rho_2 + d_4\dot{\gamma} - n^2 d_4\rho^2 + d_8\gamma - 2nd_{10}\rho$$

$$\tilde{a}_9 = d_1\dot{\rho}_1 + d_2\dot{\rho}_2 - 2n^2 d_4\rho\dot{\rho} + d_5\rho_1 + d_6\rho_2 - n^2 d_8\rho^2 + 2nd_{10}\gamma\rho$$

$$\tilde{b}_1 = -n\rho(e_1 - e_2\rho_1\delta_d) + e_3 - e_5\rho_1\delta_d$$

$$\tilde{b}_2 = -n[e_1\dot{\rho} - e_2(\rho\rho_1)^{\cdot}\delta_d] + e_4\gamma - e_5\dot{\rho}_1\delta_d - e_6\gamma\rho_1\delta_d - n\rho(e_7 - e_8\rho_1\delta_d)$$

$$\tilde{b}_3 = e_1 - e_2\rho_2\delta_d$$

$$\tilde{b}_4 = -e_1\gamma + e_2\gamma(2\rho_2 - \rho_1)\delta_d - e_2\dot{\rho}_2\delta_d + e_7 - e_8\rho_2\delta_d$$

$$\tilde{b}_5 = -e_1\dot{\gamma} + e_2\gamma(2\dot{\rho}_2 - \dot{\rho}_1)\delta_d + e_2\dot{\gamma}(2\rho_2 - \rho_1)\delta_d + n\rho(e_4 - e_6\rho_2\delta_d)$$
$$-e_7\gamma + e_8\gamma(2\rho_2 - \rho_1)\delta_d$$

$$\tilde{b}_6 = -2ne_2\rho + e_5$$

$$\tilde{b}_7 = +2ne_2(\gamma\rho - \dot{\rho}) + e_6\gamma - 2ne_8\rho$$

$$\tilde{b}_8 = +2ne_2(\gamma\rho)^{\cdot} + e_3\rho_1 + e_4\rho_2 - n^2 e_6\rho^2 + 2ne_8\gamma\rho$$

$$\tilde{c}_1 = f_1 - f_3 \rho_1 \delta_d$$

$$\tilde{c}_2 = f_2 \gamma - 3f_3 \dot{\rho}_1 \delta_d - f_4 \gamma \rho_1 \delta_d + f_5 - f_7 \rho_1 \delta_d$$

$$\tilde{c}_3 = +2f_2 \dot{\gamma} - 3f_3 \ddot{\rho}_1 \delta_d - 2f_4 (\gamma \rho_1)^{\cdot} \delta_d + f_6 \gamma - 2f_7 \dot{\rho}_1 \delta_d - f_8 \gamma \rho_1 \delta_d - n\rho(f_9 - f_{10} \rho_1 \delta_d)$$
$$+ f_{11} - f_{13} \rho_1 \delta_d$$

$$\tilde{c}_4 = f_2 \ddot{\gamma} - f_3 \dddot{\rho}_1 \delta_d - f_4 (\gamma \rho_1)^{\cdot\cdot} \delta_d + f_6 \dot{\gamma} - f_7 \ddot{\rho}_1 \delta_d - f_8 (\gamma \rho_1)^{\cdot} \delta_d - nf_9 \dot{\rho} + nf_{10} (\rho \rho_1)^{\cdot} \delta_d$$
$$+ f_{12} \gamma - f_{13} \dot{\rho}_1 \delta_d - f_{14} \gamma \rho_1 \delta_d - n\rho(f_{15} - f_{16} \rho_1 \delta_d)$$

$$\tilde{c}_5 = n\rho(f_2 - f_4 \rho_2 \delta_d) + f_9 - f_{10} \rho_2 \delta_d$$

$$\tilde{c}_6 = n[2f_2 \dot{\rho} - 2f_4 (\rho \rho_2)^{\cdot} \delta_d + f_6 \rho - f_8 \rho \rho_2 \delta_d] - f_9 \gamma + f_{10} \gamma (2\rho_2 - \rho_1) \delta_d - f_{10} \dot{\rho}_2 \delta_d$$
$$+ f_{15} - f_{16} \rho_2 \delta_d$$

$$\tilde{c}_7 = n[f_2 \ddot{\rho} - f_4 (\rho \rho_2)^{\cdot\cdot} \delta_d + f_6 \dot{\rho} - f_8 (\rho \rho_2)^{\cdot} \delta_d] - f_9 \ddot{\gamma} + f_{10} \gamma (2\dot{\rho}_2 - \dot{\rho}_1) \delta_d$$
$$+ f_{10} \dot{\gamma} (2\rho_2 - \rho_1) \delta_d + n\rho(f_{12} - f_{14} \rho_2 \delta_d) - f_{15} \gamma + f_{16} \gamma (2\rho_2 - \rho_1) \delta_d$$

$$\tilde{c}_8 = + f_3$$

$$\tilde{c}_9 = f_4 \gamma + f_7$$

$$\tilde{c}_{10} = f_1 \rho_1 + f_2 \rho_2 + f_4 (2\dot{\gamma} - n^2 \rho^2) + f_8 \gamma - 2nf_{10} \rho + f_{13}$$

$$\tilde{c}_{11} = 2f_1 \dot{\rho}_1 + 2f_2 \dot{\rho}_2 + f_4 (\ddot{\gamma} - 4n^2 \rho \dot{\rho}) + f_5 \rho_1 + f_6 \rho_2 + f_8 (\dot{\gamma} - n^2 \rho^2) + 2nf_{10} (\gamma \rho - \dot{\rho})$$
$$+ f_{14} \gamma - 2nf_{16} \rho$$

$$\tilde{c}_{12} = f_1 \ddot{\rho}_1 + f_2 \ddot{\rho}_2 - 2n^2 f_4 (\rho \dot{\rho})^{\cdot} + f_5 \dot{\rho}_1 + f_6 \dot{\rho}_2 - 2n^2 f_8 \rho \dot{\rho} + 2nf_{10} (\gamma \rho)^{\cdot} + f_{11} \rho_1$$
$$+ f_{12} \rho_2 - n^2 f_{14} \rho^2 + 2nf_{16} \gamma \rho$$

$$\bar{\bar{a}}_3 = \delta_d(\rho_1^2 N_{x0} - \delta_{ph}\rho_1 p)$$

$$\bar{\bar{a}}_8 = -\delta_d \rho_1 N_{x0} + \delta_d \delta_{ph} p$$

$$\bar{\bar{b}}_5 = \delta_d\left(\rho_2^2 N_{00} - \delta_{ph}\rho_2 p\right)$$

$$\bar{\bar{b}}_8 = \delta_d(n\rho\rho_2 N_{00} - \delta_{ph} n\rho p)$$

$$\bar{\bar{c}}_3 = -\delta_d \rho_1 N_{x0} + \delta_{ph} p$$

$$\bar{\bar{c}}_4 = \delta_d(-\rho_1 \dot{N}_{x0} - \rho_1 \gamma N_{x0} - \dot{\rho}_1 N_{x0}) + \delta_{ph}\gamma p$$

$$\bar{\bar{c}}_7 = -\delta_d n\rho\rho_2 N_{00} + \delta_{ph} n\rho p$$

$$\bar{\bar{c}}_{10} = N_{x0}$$

$$\bar{\bar{c}}_{11} = \dot{N}_{x0} + \gamma N_{x0}$$

$$\bar{\bar{c}}_{12} = -n^2\rho^2 N_{00} + \delta_{ph}(\rho_1 + \rho_2)p$$

$$\bar{a}_2 = \delta_d \delta_\omega \rho_1 d_1 \omega_{00}$$

$$\bar{a}_3 = \delta_d\left[\delta_\omega \rho_1(\gamma d_2 - \dot{\rho}_1 f_1)\omega_{00} + \delta_\omega \delta_{EL} D_{11} \rho_1^2 \omega_{00}^2\right]$$

$$\bar{a}_5 = -\delta_d \delta_\omega \rho_1 e_3 \omega_{00}$$

$$\bar{a}_7 = \delta_d \delta_\omega \rho_1 f_1 \omega_{00}$$

$$\bar{a}_8 = \delta_d\left(\delta_\omega \gamma \rho_1 f_2 \omega_{00} - \delta_\omega \delta_{EL} D_{11} \rho_1 \omega_{00}^2\right)$$

$$\bar{a}_9 = \delta_d\left[\delta_\omega \rho_1\left(\rho_1 D_{11} + \rho_2 D_{12} - n^2\rho^2 f_2\right)\omega_{\theta 0}\right]$$

$$\bar{b}_2 = -\delta_d \delta_\omega \rho_2 d_9 \omega_{00}$$

$$\bar{b}_4 = \delta_d \delta_\omega \rho_2 e_1 \omega_{00}$$

$$\bar{b}_5 = -\delta_d\delta_\omega\rho_2\gamma[D_{33} - (2\rho_2 - \rho_1)D_{36}]\omega_{00} + \delta_d\delta_{EL}\delta_\omega D_{33}\rho_2^2\omega_{00}^2$$

$$\bar{b}_7 = -\delta_d\delta_\omega\rho_2 f_9\omega_{00}$$

$$\bar{b}_8 = \delta_d\left(\delta_\omega\rho_2\gamma f_9\omega_{00} + \delta_{EL}\delta_\omega D_{33}n\rho\rho_2\omega_{\theta 0}^2\right)$$

$$\bar{c}_2 = -\delta_\omega(D_{11} - \rho_1 D_{14}\delta_d)\omega_{\theta 0}$$

$$\bar{c}_3 = -\delta_\omega[(d_5 + 2\gamma d_2 - 2\dot{\rho}_1 f_1\delta_d)\omega_{00} + d_1\dot{\omega}_{00}] - \delta_d\delta_{EL}\delta_\omega D_{11}\rho_1\omega_{00}^2$$

$$\bar{c}_4 = -\delta_\omega\Big\{\Big[\gamma d_6 - \tfrac{1}{2}\dot{\rho}_1 f_5\delta_d + \dot{\gamma}d_2 - \ddot{\rho}_1 f_1\delta_d + \gamma^2(D_{22} - \rho_1 D_{25}\delta_d) - \tfrac{3}{2}\gamma\dot{\rho}_1 f_{12}\delta_d$$
$$- n\rho d_9\Big]\omega_{00} + (\gamma d_2 - \dot{\rho}_1 f_1\delta_d)\dot{\omega}_{00}\Big\} + \delta_d\delta_\omega\delta_{EL}\Big\{[-(\dot{D}_{11} + \gamma D_{11})\rho_1$$
$$- D_{11}\dot{\rho}_1]\omega_{00}^2 - 2\rho_1 D_{11}\omega_{00}\dot{\omega}_{00}\Big\}$$

$$\bar{c}_6 = \delta_\omega(e_3 - n\rho e_1)\omega_{00}$$

$$\bar{c}_7 = -\delta_\omega\Big(\Big\{-\gamma e_3 + n\rho[\dot{D}_{12} - \rho_2\dot{D}_{15}\delta_d - \dot{\rho}_2 D_{15}\delta_d - \gamma D_{33} + \gamma(2\rho_2 - \rho_1)D_{36}\delta_d]$$
$$+ n\dot{\rho}(D_{12} - \rho_2 D_{15}\delta_d)\Big\}\omega_{00} - e_3\dot{\omega}_{00}\Big) - \delta_d\delta_{EL}\delta_\omega D_{33}n\rho\rho_2\omega_{\theta 0}^2$$

$$\bar{c}_9 = -\delta_\omega f_1\omega_{00}$$

$$\bar{c}_{10} = -\delta_\omega\Big\{[D_{14} + \gamma(D_{14} + D_{15})]\omega_{00} + D_{14}\dot{\omega}_{00} - \delta_{EL}D_{11}\omega_{00}^2\Big\}$$

$$\bar{c}_{11} = -\delta_\omega\Big\{\Big[\gamma(\dot{D}_{15} + \gamma D_{15}) + \rho_1 D_{11} + \rho_2 D_{12} + (\dot{\gamma} - n^2\rho^2)D_{15} - n\rho f_9\Big]\omega_{00}$$
$$+ \gamma f_2\dot{\omega}_{00}\Big\} + \delta_{EL}\delta_\omega\left(\dot{D}_{11}\omega_{00}^2 + 2D_{11}\omega_{00}\dot{\omega}_{\theta 0}\right)$$

$$\bar{c}_{12} = -\delta\omega\Big\{\Big[\rho_1(\dot{D}_{11} + \gamma D_{11}) + \rho_2(\dot{D}_{12} + \gamma D_{12}) + \dot{\rho}_1 D_{11} + \dot{\rho}_2 D_{12}$$
$$- n^2\rho^2(D_{15} + \gamma D_{15}) - 2n^2\rho\dot{\rho}D_{15} + n\rho\gamma f_9\Big]\omega_{00}$$
$$+ \left(\rho_1 D_{11} + \rho_2 D_{12} - n^2\rho^2 D_{15}\right)\dot{\omega}_{00}\Big\} - \delta_{EL}\delta_\omega n^2\rho^2 D_{33}\omega_{00}^2$$

$$\hat{a}_2 = -\delta_d \delta_{EL} d_1 \rho_1 \omega_{00}$$

$$\hat{a}_3 = -\delta_d \delta_{EL}[d_1(\rho_1 \dot{\omega}_{00} + \dot{\rho}_1 \omega_{00}) + d_5 \rho_1 \omega_{00}]$$

$$\hat{a}_5 = -\delta_d \delta_{EL} d_9 \rho_2 \omega_{00}$$

$$\hat{a}_7 = \delta_{EL} d_1 \omega_{00}$$

$$\hat{a}_8 = \delta_{EL}(d_1 \dot{\omega}_{00} + d_5 \omega_{00})$$

$$\hat{a}_9 = -\delta_{EL} n\rho d_9 \omega_{00}$$

$$\hat{b}_2 = -\delta_d \delta_{EL} e_3 \rho_1 \omega_{00}$$

$$\hat{b}_4 = -\delta_d \delta_{EL} e_1 \rho_2 \omega_{00}$$

$$\hat{b}_5 = -\delta_d \delta_{EL}[e_1(\rho_2 \dot{\omega}_{00} + \dot{\rho}_2 \omega_{00}) + e_7 \rho_2 \omega_{00}]$$

$$\hat{b}_7 = -\delta_{EL}(e_1 n\rho - e_3)\omega_{00}$$

$$\hat{b}_8 = -n\delta_{EL}[e_1(\rho\dot{\omega}_{00} + \dot{\rho}\omega_{00}) + \rho e_7 \omega_{00}]$$

$$\hat{c}_2 = -f_1 \rho_1 \delta_{EL} \omega_{00} \delta_d$$

$$\hat{c}_3 = -\delta_d \delta_{EL}[2f_1(\rho_1 \dot{\omega}_{00} + \dot{\rho}_1 \omega_{00}) + f_5 \rho_1 \omega_{00}]$$

$$\hat{c}_4 = -\delta_d \delta_{EL}[f_1(\rho_1 \ddot{\omega}_{00} + 2\dot{\rho}_1 \dot{\omega}_{00} + \ddot{\rho}_1 \omega_{00}) + f_5(\rho_1 \dot{\omega}_{00} + \dot{\rho}_1 \omega_{00}) + \rho_1 f_{11} \omega_{00}]$$

$$\hat{c}_6 = -\delta_d f_9 \rho_2 \delta_{EL} \omega_{00}$$

$$\hat{c}_7 = -\delta_d \delta_{EL}[f_9(\rho_2 \dot{\omega}_{00} + \dot{\rho}_2 \omega_{00}) + \rho_2 f_{15} \omega_{00}]$$

$$\hat{c}_9 = + f_1 \delta_{EL} \omega_{00}$$

$$\hat{c}_{10} = + \delta_{EL}(2f_1 \dot{\omega}_{00} + f_5 \omega_{00})$$

$$\hat{c}_{11} = + \delta_{EL}(f_1 \ddot{\omega}_{00} + f_5 \dot{\omega}_{00} - n\rho f_9 \omega_{00} + f_{11} \omega_{00})$$

$$\hat{c}_{12} = -n\delta_{EL}[f_9(\rho\dot{\omega}_{00} + \dot{\rho}\omega_{00}) + \rho f_{15} \omega_{00}]$$

$$\overset{*}{a}_3 = \delta_d \delta_{RN} n^2 \rho^2 N_{00}$$

$$\overset{*}{b}_2 = \delta_d \delta_{RN} n \rho \gamma N_{00}$$

$$\overset{*}{b}_3 = -\delta_d \delta_{RN} N_{x0}$$

$$\overset{*}{b}_4 = -\delta_d \delta_{RN} \dot{N}_{x0}$$

$$\overset{*}{b}_5 = \delta_d \delta_{RN} [\gamma(\gamma N_{x0} + \dot{N}_{x0}) + \dot{\gamma} N_{x0}]$$

$$d_1 = D_{11} - \rho_1 D_{14} \delta_d$$

$$d_2 = D_{12} - \rho_1 D_{15} \delta_d$$

$$d_3 = D_{14} - \rho_1 D_{44} \delta_d$$

$$d_4 = D_{15} - \rho_1 D_{45} \delta_d$$

$$d_5 = \dot{D}_{11} - \rho_1 \dot{D}_{14} \delta_d + \gamma[(D_{11} - D_{12}) - \rho_1(D_{14} - D_{15})\delta_d]$$

$$d_6 = \dot{D}_{12} - \rho_1 \dot{D}_{15} \delta_d + \gamma[(D_{12} - D_{22}) - \rho_1(D_{15} - D_{25})\delta_d]$$

$$d_7 = \dot{D}_{14} - \rho_1 \dot{D}_{44} \delta_d + \gamma[(D_{14} - D_{15}) - \rho_1(D_{44} - D_{45})\delta_d]$$

$$d_8 = \dot{D}_{15} - \rho_1 \dot{D}_{45} \delta_d + \gamma[(D_{15} - D_{25}) - \rho_1(D_{45} - D_{55})\delta_d]$$

$$d_9 = n\rho(D_{33} - \rho_1 D_{36} \delta_d)$$

$$d_{10} = n\rho(D_{36} - \rho_1 D_{66} \delta_d)$$

$$e_1 = D_{33} - \rho_2 D_{36} \delta_d$$

$$e_2 = D_{36} - \rho_2 D_{66} \delta_d$$

$$e_3 = -n\rho(D_{12} - \rho_2 D_{15} \delta_d)$$

$$e_4 = -n\rho(D_{22} - \rho_2 D_{25} \delta_d)$$

$$e_5 = -n\rho(D_{15} - \rho_2 D_{45} \delta_d)$$

$$e_6 = -n\rho(D_{25} - \rho_2 D_{55}\delta_d)$$

$$e_7 = \dot{D}_{33} - \rho_2\dot{D}_{36}\delta_d + 2\gamma(D_{33} - \rho_2 D_{36}\delta_d)$$

$$e_8 = \dot{D}_{36} - \rho_2\dot{D}_{66}\delta_d + 2\gamma(D_{36} - \rho_2 D_{66}\delta_d)$$

$$f_1 = D_{14}$$

$$f_2 = D_{15}$$

$$f_3 = D_{44}$$

$$f_4 = D_{45}$$

$$f_5 = 2\dot{D}_{14} + \gamma(2D_{14} - D_{15})$$

$$f_6 = 2\dot{D}_{15} + \gamma(2D_{15} - D_{25})$$

$$f_7 = 2\dot{D}_{44} + \gamma(2D_{44} - D_{45})$$

$$f_8 = 2\dot{D}_{45} + \gamma(2D_{45} - D_{55})$$

$$f_9 = 2n\rho D_{36}$$

$$f_{10} = 2n\rho D_{66}$$

$$f_{11} = \ddot{D}_{14} + \gamma(2\dot{D}_{14} - \dot{D}_{15}) + \rho_2 D_{12} + \rho_1 D_{11} - n^2\rho^2 D_{15} - \rho_1\rho_2(D_{14} - D_{15})$$

$$f_{12} = \ddot{D}_{15} + \gamma(2\dot{D}_{15} - \dot{D}_{25}) + \rho_2 D_{22} + \rho_1 D_{12} - n^2\rho^2 D_{25} - \rho_1\rho_2(D_{15} - D_{25})$$

$$f_{13} = \ddot{D}_{44} + \gamma(2\dot{D}_{44} - \dot{D}_{45}) + \rho_2 D_{15} + \rho_1 D_{14} - n^2\rho^2 D_{45} - \rho_1\rho_2(D_{44} - D_{45})$$

$$f_{14} = \ddot{D}_{45} + \gamma(2\dot{D}_{45} - \dot{D}_{55}) + \rho_2 D_{25} + \rho_1 D_{15} - n^2\rho^2 D_{55} - \rho_1\rho_2(D_{45} - D_{55})$$

$$f_{15} = 2n\rho(\dot{D}_{36} + \gamma D_{36})$$

$$f_{16} = 2n\rho(\dot{D}_{66} + \gamma D_{66})$$

A3 Boundary Conditions

The equations governing the stability of a shell of revolution have been expressed in terms of the displacement components u, v, and w. Hence, it is also necessary to express the natural boundary conditions ($\bar{N}_{x\theta}$, $\bar{N}_x$, $\bar{Q}_x$, $\bar{M}_x$) in terms of the displacements. This is effected through use of the following equations:

$$\bar{N}_{x\theta} = N_{x\theta} - \delta_d \rho_2 M_{x\theta} = i_2 v + i_4 \dot{u} + i_5 u + i_7 \dot{w} + i_8 w$$

$$\bar{N}_x = j_2 \dot{v} + j_3 v + j_5 u + j_7 \ddot{w} + j_8 \dot{w} + j_9 w$$

$$\bar{Q}_x = \dot{M}_x + \gamma(M_x - M_\theta) + 2n\rho M_{x\theta} + N_{x0}\omega_\theta + \delta_{BC} N_x \omega_{00}$$

$$= k_2 \ddot{v} + k_3 \dot{v} + k_4 v + k_6 \dot{u} + k_7 u + k_9 \dddot{w} + k_{10} \ddot{w} + k_{11} \dot{w} + k_{12} w$$

$$\bar{M}_x = \ell_1 \dot{v} + \ell_2 v + \ell_3 u + \ell_4 \ddot{w} + \ell_5 \dot{w} + \ell_6 w$$

where

$$i_2 = n\rho(-e_1 + \delta_d \rho_1 e_2)$$

$$i_4 = e_1 - \delta_d \rho_2 e_2$$

$$i_5 = \gamma[-e_1 + \delta_d e_2(2\rho_2 - \rho_1)] + \delta_{EL} \rho_2 e_1 \omega_{00}$$

$$i_7 = -2n\rho e_2$$

$$i_8 = +2n\gamma\rho e_2 + \delta_{EL} n\rho e_1 \omega_{\theta 0}$$

$$j_2 = D_{11} - \delta_d \rho_1 D_{14}$$

$$j_3 = \gamma d_2 - \delta_d \dot{\rho}_1 f_1 + \delta_d \delta_{EL} \rho_1 D_{11} \omega_{00}$$

$$j_5 = n\rho(D_{12} - \delta_d \rho_2 D_{15})$$

$$j_7 = f_1$$

$$j_8 = \gamma f_2 - \delta_d \delta_{EL} D_{11} \omega_{00}$$

$$j_9 = \rho_1 D_{11} + \rho_2 D_{12} - n^2 \rho^2 f_2$$

$$k_2 = f_1 - \rho_1 f_3 \delta_d$$

$$k_3 = \frac{1}{2}(f_5 - \gamma f_2) - \frac{1}{2}\rho_1(f_7 - \gamma f_4)\delta_d + \gamma f_2 - 2\dot{\rho}_1 f_3 \delta_d - \gamma\rho_1 f_4 \delta_d$$
$$+ (\rho_1 f_1 \delta_d \delta_{EL} + d_1 \delta_{BC})\omega_{\theta 0}$$

$$k_4 = \frac{1}{2}\gamma(f_6 - \gamma D_{25}) - \frac{1}{2}\dot{\rho}_1(f_7 - \gamma f_4)\delta_d - \frac{1}{2}\gamma\rho_1(f_8 - \gamma D_{55})\delta_d + \dot{\gamma} f_2 - \ddot{\rho}_1 f_3 \delta_d$$
$$- (\gamma\rho_1)^{\cdot} f_4 \delta_d - n\rho(f_9 - \rho_1 f_{10}\delta_d) + \Big\{\delta_{EL}\Big[\frac{1}{2}(f_5 - \gamma f_2)\rho_1 \delta_d \omega_{\theta 0} + f_1 \delta_d(\rho_1 \dot{\omega}_{\theta 0}$$
$$+ \dot{\rho}_1 \omega_{\theta 0})\Big] + \delta_{BC}(\gamma d_2 - \dot{\rho}_1 D_{14}\delta_d)\omega_{\theta 0}\Big\} + \rho_1 N_{x0}\delta_d + \rho_1 D_{11}\delta_{EL}\delta_{BC}\delta_d \omega_{\theta 0}^2$$

$$k_6 = n\rho(f_2 - \rho_2 f_4 \delta_d) + (f_9 - \rho_2 f_{10}\delta_d)$$

$$k_7 = n\rho\Big[\frac{1}{2}(f_6 - \gamma D_{25}) - \frac{1}{2}\rho_2(f_8 - \gamma D_{55})\delta_d\Big] + n\dot{\rho} f_2 - n(\rho\rho_2)^{\cdot} f_4 \delta_d$$
$$+ \gamma\Big[-f_9 + (2\rho_2 - \rho_1)f_{10}\delta_d\Big] + (f_9 \rho_2 \delta_{EL}\delta_d - e_3 \delta_{BC})\omega_{\theta 0}$$

$$k_9 = f_3$$

$$k_{10} = \frac{1}{2}(f_7 - \gamma f_4) + \gamma f_4 - f_1(\delta_{EL} - \delta_{BC})\omega_{\theta 0}$$

$$k_{11} = \frac{1}{2}\gamma(f_8 - \gamma D_{55}) + \rho_1 f_1 + \rho_2 f_2 + (\dot{\gamma} - n^2\rho^2)f_4 - 2n\rho f_{10} - \delta_{EL}\Big[\frac{1}{2}(f_5 - \gamma f_2)\omega_{\theta 0}$$
$$+ f_1 \dot{\omega}_{\theta 0}\Big] + \gamma f_2 \delta_{BC}\omega_{\theta 0} - N_{x0} - D_{11}\delta_{EL}\delta_{BC}\omega_{\theta 0}^2$$

$$k_{12} = \frac{1}{2}\rho_1(f_5 - \gamma f_2) + \frac{1}{2}\rho_2(f_6 - \gamma D_{25}) - \frac{1}{2}n^2\rho^2(f_8 - \gamma D_{55}) + \dot{\rho}_1 f_1 + \dot{\rho}_2 f_2$$
$$- 2n^2\rho\dot{\rho} f_4 + 2n\gamma\rho f_{10} + \Big[n\rho f_9 \delta_{EL} + \delta_{BC}(\rho_1 D_{11} + \rho_2 D_{12} - n^2\rho^2 D_{15})\Big]\omega_{\theta 0}$$

$$\ell_1 = f_1 - \delta_d \rho_1 f_3$$

$$\ell_2 = -\delta_d f_3 \dot{\rho}_1 + \gamma(f_2 - \delta_d \rho_1 f_4) + \delta_d \delta_{EL}\rho_1 f_1 \omega_{\theta 0}$$

$$\ell_3 = n\rho(f_2 - \delta_d \rho_2 f_4)$$

$$\ell_4 = f_3$$

$$\ell_5 = \gamma f_4 - \delta_{EL} f_1 \omega_{00}$$

$$\ell_6 = \rho_1 f_1 + \rho_2 \rho_2 - n^2 \rho^2 f_4$$

It is often convenient to represent the boundary conditions in terms of axial and radial components instead of tangential and normal components. The relations between these two representations are as follows:

$$H = \overline{N}_x \cos \phi^* - \overline{Q}_x \sin \phi^*$$

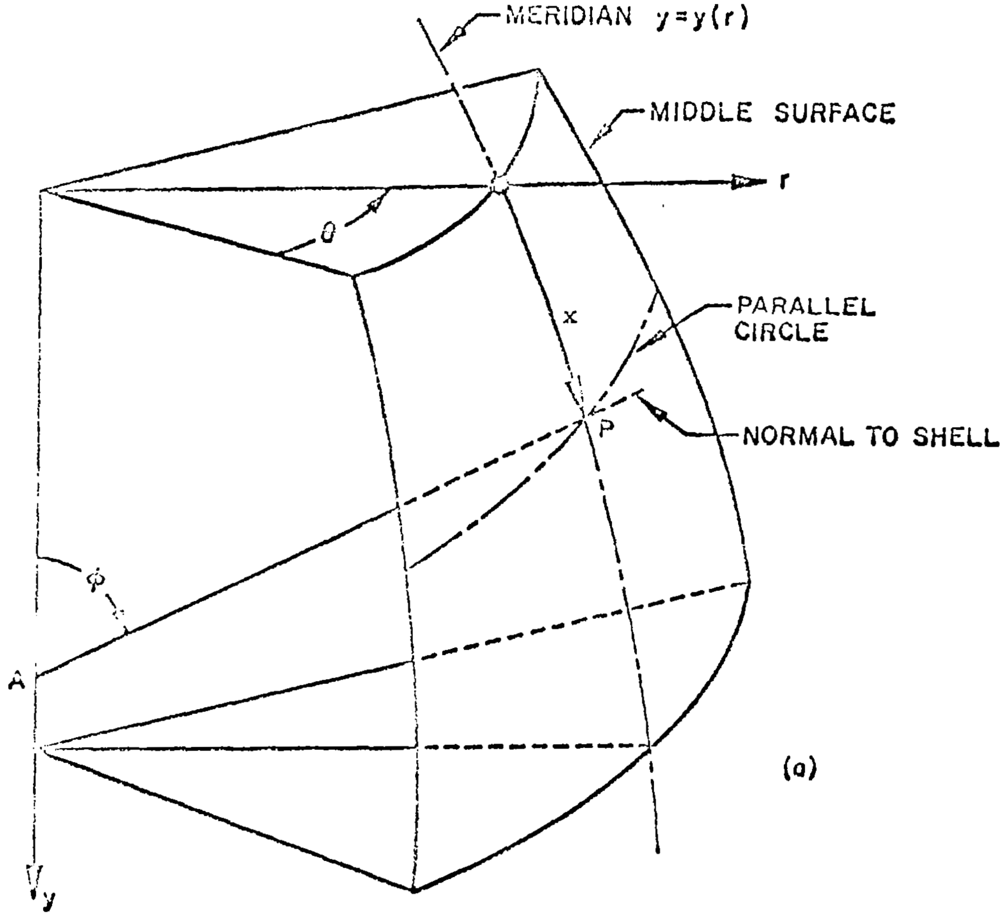
$$V = \overline{N}_x \sin \phi^* + \overline{Q}_x \cos \phi^*$$

$$u_H = v \cos \phi + w \sin \phi$$

$$u_V = v \sin \phi - w \cos \phi$$

where

$$\phi^* = \phi + \omega_{00}$$



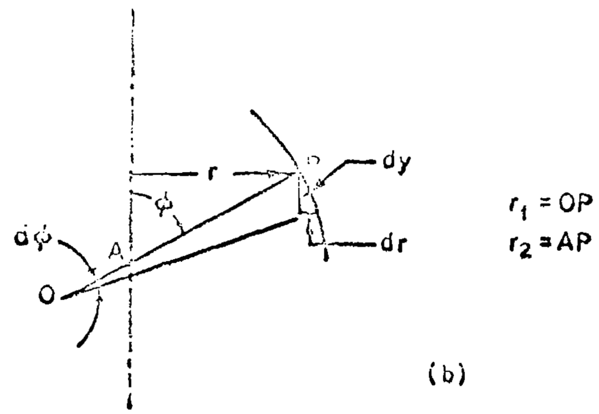


Fig. 7 Notation for a Shell of Revolution

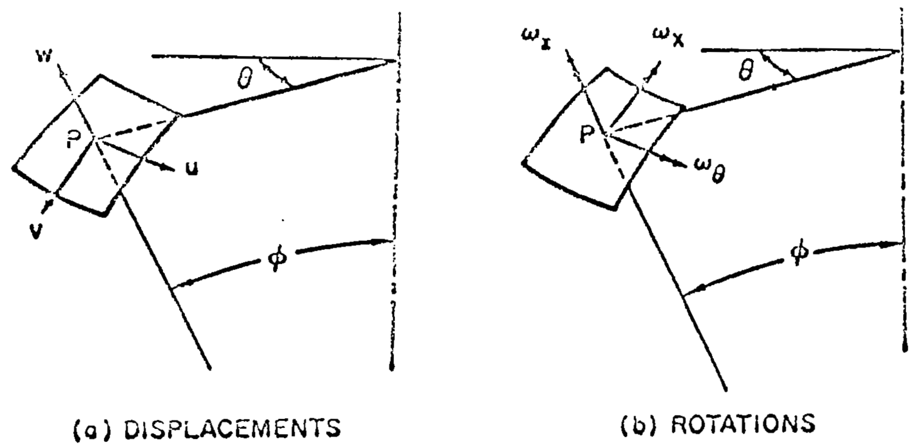


Fig. 8 Displacement and Rotation Components

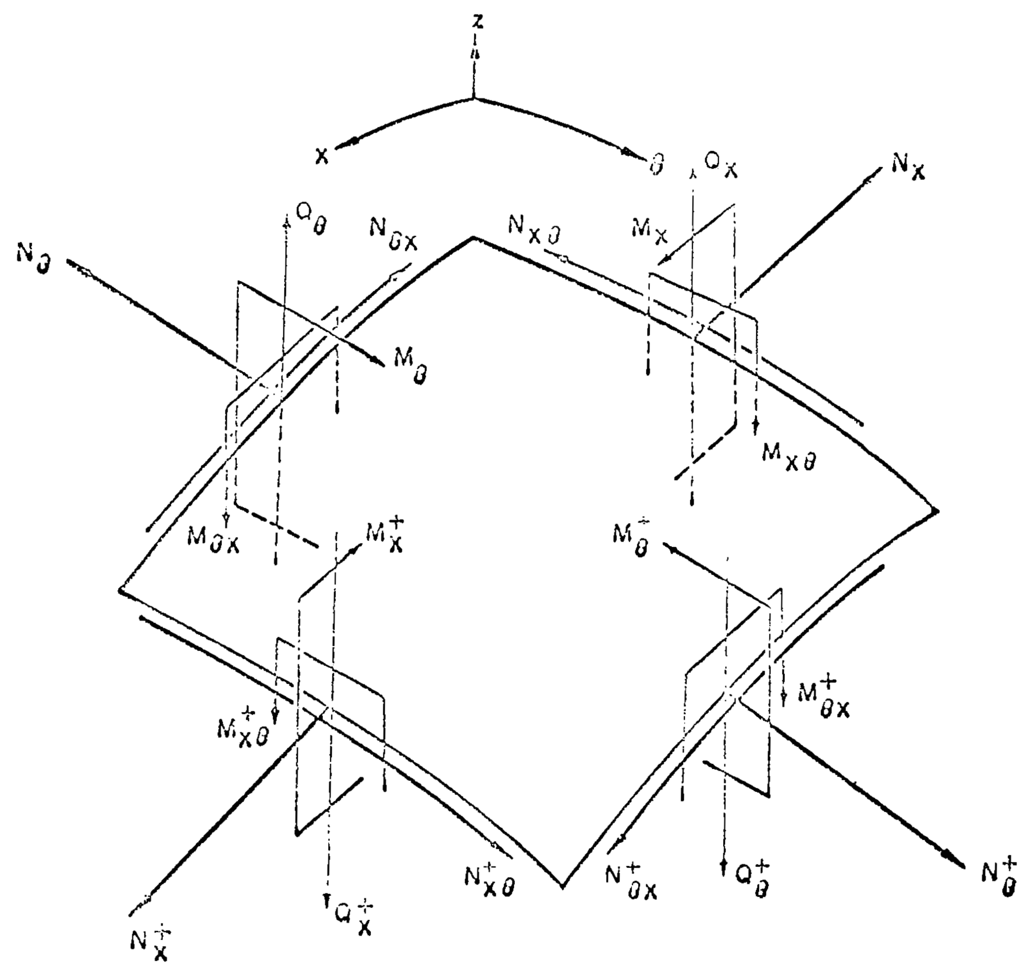


Fig. 9 Stress Resultants

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